

A TREATISE ON THE DIFFERENTIAL GEOMETRY OF CURVES AND SURFACES

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GINN AND COMPANY

BOSTON NEW YORK CHICAGO LONDON



The Athenaeum Press
GINN AND COMPANY PRO
PRIETORS BOSTON U S A

89 8

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PREFACE

book is a development from courses which I have given in 1 for a number of years. During this time I have come to more would be accomplished by my students if they had an ory treatise written in English and otherwise adapted to the en beginning their graduate work.

r I is devoted to the theory of twisted curves, the method d being that which is usually followed in discussions of this. But in addition I have introduced the idea of moving axes, derived the formulas pertaining thereto from the previously Frenet-Serret formulas. In this way the student is made with a method which is similar to that used by Darboux in volume of his *Leçons*, and to that of Cesaro in his *Geometria a*. This method is not only of great advantage in the treat-certain topics and in the solution of problems, but it is valu-eveloping geometrical thinking.

remainder of the book may be divided into three parts. The sisting of Chapters II–VI, deals with the geometry of a sur-e neighborhood of a point and the developments therefrom, curves and systems of curves defined by differential equa-; a large extent the method is that of Gauss, by which the s of a surface are derived from the discussion of two quad-erential forms. However, little or no space is given to the treatment of differential forms and their invariants. In the method of moving axes, as defined in the first chapter, extended so as to be applicable to an investigation of the s of surfaces and groups of surfaces. The extent of the ncerning ordinary points is so great that no attempt has le to consider the exceptional problems. For a discussion questions as the existence of integrals of differential equa-l boundary conditions the reader must consult the treatises al particularly with these subjects.

apters VII and VIII the theory previously developed is o several groups of surfaces, such as the quadrics, ruled minimal surfaces, surfaces of constant total curvature, and with plane and spherical lines of curvature.

The idea of applicability of surfaces is introduced in Chapter III as a particular case of conformal representation, and throughout the book attention is called to examples of applicable surfaces. However, the general problems concerned with the applicability of surfaces are discussed in Chapters IX and X, the latter of which deals entirely with the recent method of Weingarten and its developments. The remaining four chapters are devoted to a discussion of infinitesimal deformation of surfaces, congruences of straight lines and of circles, and triply orthogonal systems of surfaces.

It will be noticed that the book contains many examples, and the student will find that whereas certain of them are merely direct applications of the formulas, others constitute extensions of the theory which might properly be included as portions of a more extensive treatise. At first I felt constrained to give such references as would enable the reader to consult the journals and treatises from which some of these problems were taken, but finally it seemed best to furnish no such key, only to remark that the *Encyklopadie der mathematischen Wissenschaften* may be of assistance. And the same may be said about references to the sources of the subject-matter of the book. Many important citations have been made, but there has not been an attempt to give every reference. However, I desire to acknowledge my indebtedness to the treatises of Darboux, Bianchi, and Scheffers. But the difficulty is that for many years I have consulted these authors so freely that now it is impossible for me to say, except in certain cases, what specific debts I owe to each.

In its present form, the material of the first eight chapters has been given to beginning classes in each of the last two years, and the remainder of the book, with certain enlargements, has constituted an advanced course which has been followed several times. It is impossible for me to give suitable credit for the suggestions made and the assistance rendered by my students during these years, but I am conscious of helpful suggestions made by my colleagues, Professors Veblen, MacInnes, and Swift, and by my former colleague, Professor Bliss of Chicago. I wish also to thank Mr. A. K. Krause for making the drawings for the figures.

It remains for me to express my appreciation of the courtesy shown by Ginn and Company, and of the assistance given by them during the printing of this book.

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DIFFERENTIAL GEOMETRY

CHAPTER I

CURVES IN SPACE

Parametric equations of a curve Consider space referred to rectangular axes, and let (x, y, z) denote as usual the coordinates of a point with respect to these axes. In the plane $z=0$ consider a circle of radius r and center (a, b) . The coordinates of a point P on the circle can be expressed in the form

$$x = a + r \cos u, \quad y = b + r \sin u, \quad z = 0,$$

where u denotes the angle which the radius to P makes with the x -axis. As u varies from 0° to 360° , the point P describes the circle. The quantities a, b, r determine the position and size of the circle, whereas u determines the position of a point upon it. In this sense it is a *variable* or *parameter* for the circle. And equations (1) are called *parametric equations* of the circle.

A straight line in space is determined by a point $P_0(a, b, c)$, and its direction cosines

α, β, γ . The latter fix also the sense of the line. Choose another point on the line, and let the distance P_0P be denoted by u , which is positive or negative.

The rectangular coordinates of P are then expressible in the form

$$x = a + u\alpha, \quad y = b + u\beta, \quad z = c + u\gamma$$

For every value of u there corresponds a point on the line, and the coordinates of any point on the line are expressible as in (2). These equations are consequently *parametric equations* of the straight line.

As in fig 1, a line segment PD , of constant length a , perpendicular to a line OZ at D , revolves uniformly about OZ as axis,

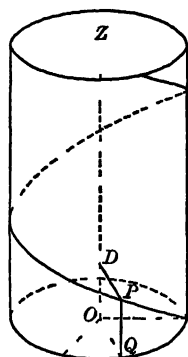


FIG 1

ident that the values of x, y, z , given by (7) for a value u equal to those given by (4) for the corresponding value x obtained from (6). Consequently equations (4) and (7) define the curve, u and v being the respective parameters. Since there is no restriction upon the function ϕ , except that it be analytic, it follows that a curve can be given parametric representation in an infinity of ways.

other forms of the equations of a curve. If the first of equations (4) be solved for u , giving $u = \phi(x)$, then, in terms of x as the parameter, equations (7) are

$$x = x, \quad y = F_2(x), \quad z = F_3(x)$$

where the curve is really defined by the last two equations, and is therefore a plane curve in the xy -plane, its equation in the xy -plane being

$$y = f(x)$$

points in space whose coordinates satisfy the equation (8) lie on the cylinder whose elements are parallel to the z -axis and whose cross section by the xy -plane is the curve $y = F_2(x)$. In a similar manner, the equation $z = F_3(x)$ defines a cylinder whose elements are parallel to the y -axis. Hence the curve with the equation (8) is the locus of points common to two cylinders whose axes are perpendicular. Conversely, if lines are drawn through the points of a space curve normal to two planes perpendicular to each other, we obtain two such cylinders whose intersection is the given curve. Hence equations (8) furnish a perfectly general method of representation of a space curve.

Generally, the parameter u can be eliminated from equations (4) in such a way that there result two equations, each of which involves only two of the three rectangular coordinates. Thus,

$$\Phi_1(x, y, z) = 0, \quad \Phi_2(x, y, z) = 0$$

where, if two equations of this kind be solved for y and z as functions of x , we get equations of the form (8), and, in turn, of the form (4), by replacing x by an arbitrary function of u . Hence equations (10) also are the general equations of a curve. It will be noted later that each of these equations defines a surface.

Unless f'_1, f'_2, f'_3 be zero and the curve reduce to a point, at least one of the coordinates must be imaginary. For this case s is zero. Hence these imaginary curves are called *curves of length zero*, or *minimal curves*. For the present they will be excluded from the discussion.

Let the arc be the parameter of a given curve and s and $s + e$ its values for two points $M(x, y, z)$ and $M_1(x_1, y_1, z_1)$. By Taylor's theorem we have

$$(17) \quad \begin{cases} x_1 = x + x'e + x''\frac{e^2}{2} + \dots, \\ y_1 = y + y'e + y''\frac{e^2}{2} + \dots, \\ z_1 = z + z'e + z''\frac{e^2}{2} + \dots, \end{cases}$$

where an accent indicates differentiation with respect to s .

Unless x', y', z' are all zero, that is, unless the locus is a point and not a curve, one at least of the lengths $x_1 - x, y_1 - y, z_1 - z$ is of the order of magnitude of e . If these lengths be denoted by $\delta x, \delta y, \delta z$, and e by δs , then we have

$$\sqrt{\delta x^2 + \delta y^2 + \delta z^2} = \delta s + l_2,$$

where l_2 denotes the aggregate of terms of the second and higher orders in δs . Hence, as M_1 approaches M the ratio of the length of the chord and the arc MM_1 approaches unity, and in the limit we have $ds^2 = dx^2 + dy^2 + dz^2$.

4 Tangent to a curve The *tangent to a curve* at a point M is the limiting position of the secant through M and a point M_1 of the curve as the latter approaches M as a limit.

In order to find the equation of the tangent we take s for parameter and write the expressions for the coordinates of M_1 in the form (17). The equations of the secant through M and M_1 are

$$\frac{X-x}{x_1-x} = \frac{Y-y}{y_1-y} = \frac{Z-z}{z_1-z}$$

If each member of these equations be multiplied by e and the denominators be replaced by their values from (17), we have in the limit as M_1 approaches M

$$(18) \quad \frac{X-x}{x'} = \frac{Y-y}{y'} = \frac{Z-z}{z'}$$

tangent

β, γ denote the direction-cosines of the tangent in consequence of (15), we may take

$$\alpha = x', \quad \beta = y', \quad \gamma = z'$$

if the parameter u is any whatever, these equations are *

$$\frac{f'_1}{\sqrt{f'^2_1 + f'^2_2 + f'^2_3}}, \quad \beta = \frac{f'_2}{\sqrt{f'^2_1 + f'^2_2 + f'^2_3}}, \quad \gamma = \frac{f'_3}{\sqrt{f'^2_1 + f'^2_2 + f'^2_3}}$$

may also be written thus

$$\alpha = \frac{dx}{ds}, \quad \beta = \frac{dy}{ds}, \quad \gamma = \frac{dz}{ds}$$

these equations it follows that, if the convention be made, the positive direction on the curve is that in which the parameter increases, the positive direction upon the tangent is the direction upon the curve

A fundamental property of the tangent is discovered by considering the expression for the distance from the point M_1 , with coordinates (17), to any line through M . We write the equation of such a line in the form

$$\frac{X-x}{a} = \frac{Y-y}{b} = \frac{Z-z}{c},$$

where a, b, c are the direction-cosines

the distance from M_1 to this line is equal to

$$\left\{ [(bx' - ay')e + \frac{1}{2}(bz'' - ay'')]e^2 + \right. \\ \left. + [(cy' - bz')e + \frac{1}{2}(cz'' - bz'')]e^2 + [(az' - cx')e + \frac{1}{2}(az'' - cx'')]e^2 \right\}^{\frac{1}{2}}$$

if MM_1 be considered an infinitesimal of the first order, the distance also is of the first order unless

$$\frac{a}{x'} = \frac{b}{y'} = \frac{c}{z'},$$

in which case it is of the second order at least. But when these conditions are satisfied, equations (22) define the tangent at M . Therefore, of all the lines through a point of a curve the tangent is the one most nearly perpendicular to the curve.

Whenever the functions x, y, z appear in a formula it is understood that the arc s is the parameter, unless otherwise we use f_1, f_2, f_3 indicating by accents derivatives with respect to the parameter u .

5 Order of contact Normal plane When the curve is such that there are points for which

$$(24) \quad \frac{x''}{x'} = \frac{y''}{y'} = \frac{z''}{z'},$$

the distance from M_1 to the tangent is of the third order at least. In this case the tangent is said to have *contact of the second order* whereas, ordinarily, the *contact is of the first order*. And, in general, the tangent to a curve has contact of the n th order at a point if the following conditions are satisfied for $n = 2, \dots, n-1$, and

$$(25) \quad \frac{x^{(n)}}{x^{(n-1)}} = \frac{y^{(n)}}{y^{(n-1)}} = \frac{z^{(n)}}{z^{(n-1)}}$$

When the parameter of the curve is any whatever, equations (24), (25) are reducible to the respective equations

$$\frac{f_1''}{f_1'} = \frac{f_2''}{f_2'} = \frac{f_3''}{f_3'}, \quad \frac{f_1^{(n)}}{f_1^{(n-1)}} = \frac{f_2^{(n)}}{f_2^{(n-1)}} = \frac{f_3^{(n)}}{f_3^{(n-1)}}$$

Normal Plane The plane normal to the tangent to a curve at the point of contact is called the *normal plane* at the point. Its equation

$$(26) \quad (X-x)\alpha + (Y-y)\beta + (Z-z)\gamma = 0,$$

where α, β, γ have the values (20)

EXAMPLES

1 Put the equations of the circular helix (3) in the form (8)

2 Express the equations of the circular helix in terms of the arc measured from a point of the curve, and show that the tangents to the curve meet the elements of the circular cylinder under constant angle

3 Show that if at every point of a curve the tangency is of the second order the curve is a straight line

4 Prove that a necessary and sufficient condition that at the point (x_0, y_0) the plane curve $y = f(x)$ the tangent has contact of the n th order is $f'(x_0) = f''(x_0) = \dots = f^{(n-1)}(x_0) = 0$, also, that according as n is even or odd the tangent crosses the curve at the point or does not

5 Prove the following properties of the twisted cubic

(a) Of all the planes through a point of the cubic one and only one meet the cubic in three coincident points, its equation is $3u^2x - 3uy + z - u^3 = 0$

(b) There are no double points, but the orthogonal projection on a plane is a curve with a double point

(c) Four planes determined by a variable chord of the cubic and by each of four fixed points of the curve are in constant cross-ratio

6 Curvature Radius of first curvature. Let M, M_1 be two points of a curve, Δs the length of the arc between these points, and $\Delta\theta$ the angle between the tangents. The limiting value of $\Delta\theta/\Delta s$ as M_1 approaches M , namely $d\theta/ds$, measures the rate of change of the direction of the tangent at M as the point of contact moves along the curve. This limiting value is called the *first curvature* of the curve at M , and its reciprocal the *radius of first curvature*, the latter will be denoted by ρ .

In order to find an expression for ρ in terms of the quantities defining the curve, we introduce the idea of spherical representation as follows. We take the sphere* of unit radius with center at the origin and draw radii parallel to the positive directions of the tangents to the curve, or such a portion of it that no two tangents are parallel. The locus of the extremities is a curve upon the sphere, which is in one-to-one correspondence with the given curve. In this sense we have a *spherical representation*, or *spherical indicatrix*, of the curve.

The angle $\Delta\theta$ between the tangents to the curve at the points M, M_1 is measured by the arc of the great circle between their representative points m, m_1 on the sphere. If $\Delta\sigma$ denotes the length of the arc of the spherical indicatrix between m and m_1 , then by the result at the close of § 3,

$$\frac{d\theta}{d\sigma} = \lim \frac{\Delta\theta}{\Delta\sigma} = 1$$

Hence we have

$$27) \quad \frac{1}{\rho} = \frac{d\sigma}{ds},$$

where $d\sigma$ is the linear element of the spherical indicatrix.

The coordinates of m are the direction cosines α, β, γ of the tangent at M , consequently

$$27') \quad \frac{1}{\rho^2} = \left(\frac{d\sigma}{ds}\right)^2 = \left(\frac{d\alpha}{ds}\right)^2 + \left(\frac{d\beta}{ds}\right)^2 + \left(\frac{d\gamma}{ds}\right)^2$$

When the arc s is the parameter, this formula becomes

$$28) \quad \frac{1}{\rho^2} = x'^2 + y'^2 + z'^2$$

* Hereafter we refer to this as the *unit sphere*

Curvature

However, when the parameter is any whatever, u , we have from (12), (13), (20),

$$(29) \quad \frac{d\alpha}{ds} = \frac{1}{\phi'} \frac{d}{du} \left(\frac{f_1'}{\phi'} \right) = \frac{1}{\phi'^3} (\phi' f_1'' - \phi'' f_1'),$$

and
$$\phi' \phi'' = f_1' f_1'' + f_2' f_2'' + f_3' f_3''$$

Hence we find by substitution

$$(30) \quad \frac{1}{\rho^2} = \frac{f_1''^2 + f_2''^2 + f_3''^2 - \phi''^2}{\phi'^4},$$

which sometimes is written thus

$$(31) \quad \frac{1}{\rho^2} = \frac{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2}{ds^4}$$

$$\phi' = \sqrt{f_1'^2 + f_2'^2 + f_3'^2}$$

$$\phi'' = \frac{f_1' f_1'' + f_2' f_2'' + f_3' f_3''}{\sqrt{f_1'^2 + f_2'^2 + f_3'^2}}$$

The sign of ρ is not determined by these formulas. We make the convention that it is always positive and thus fix the sense of a displacement on the spherical indicatrix.

7 Osculating plane Consider the plane through the tangent to a curve at a point M and through a point M_1 of the curve. The limiting position of this plane as M_1 approaches M is called the *osculating plane at M* . In deriving its equation and thus establishing its existence we assume that the arc s is the parameter, and take the coordinates of M_1 in the form (17)

The equation of a plane through $M(x, y, z)$ is of the form

$$(32) \quad (X-x)a + (Y-y)b + (Z-z)c = 0,$$

X, Y, Z being the current coordinates. When the plane passes through the tangent at M , the coefficients a, b, c are such that

$$(33) \quad x'a + y'b + z'c = 0$$

If the values (17) for x_1, y_1, z_1 be substituted in (32) for X, Y, Z , and the resulting equation be divided by $\frac{e^2}{2}$, we get

$$(x''a + y''b + z''c) + \eta = 0,$$

where η represents the aggregate of the terms of first and higher orders in e . As M_1 approaches M , η approaches zero, and in the limit we have

$$(34) \quad x''a + y''b + z''c = 0$$

finding a, b, c from equations (32), (33), (34) we obtain, as the equation of the osculating plane,

$$\begin{vmatrix} X-x & Y-y & Z-z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0$$

Oscul.
p.

thus we find that when the curve is defined by equations (4) of a general parameter u , the equation of the osculating

$$\begin{vmatrix} X-x & Y-y & Z-z \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = 0$$

plane defined by either of these equations is unique except where the tangent at the point has contact of an order higher than

one. In the latter case equations (33), (34) are not independent, as from (24), and if the contact of the tangent is of the n th order the equations

$$x^{(r)}a + y^{(r)}b + z^{(r)}c = 0,$$

for values of r up to and including n are not independent of each other. But for $r = n + 1$, this equation and (33) are independent, and we have as the equation of the osculating plane at this point,

$$\begin{vmatrix} X-x & Y-y & Z-z \\ x' & y' & z' \\ x^{(n+1)} & y^{(n+1)} & z^{(n+1)} \end{vmatrix} = 0$$

if a curve is plane, and its plane is taken for the xy -plane, equation (35) reduces to $Z = 0$. Hence the osculating plane of a plane curve is the plane of the latter, and consequently is the same for all points of the curve. Conversely, when the osculating plane of a curve is the same for all its points, the curve is plane, and all the points of the curve lie in the fixed osculating plane.

The equation of the osculating plane of the twisted cubic (§ 2) is readily found to

$$3u^2X - 3uY + Z - u^3 = 0,$$

where Y, Z are current coordinates. From the definition of the osculating plane it follows that the curve is a cubic, it follows that the osculating plane meets the curve only at the point of osculation. As equation (1) is a cubic in u , it follows that through a point (x_0, y_0, z_0) not on the curve there pass three planes which are tangent to the cubic. Let u_1, u_2, u_3 denote the parameter values of these points of osculation. In (1) we have

$$u_1 + u_2 + u_3 = 3x_0, \quad u_1u_2 + u_2u_3 + u_3u_1 = 3y_0, \quad u_1u_2u_3 = z_0$$

By means of these relations the equation of the plane through the corresponding three points on the cubic is reducible to

$$(X - x_0)3y_0 - (Y - y_0)3x_0 + (Z - z_0) = 0$$

This plane passes through the point (x_0, y_0, z_0) , hence we have the theorems

The points of contact of the three osculating planes of a twisted cubic through a point not on the curve lie in a plane through the point

The osculating planes at three points of a twisted cubic meet in a point which lies in the plane of the three points

By means of these theorems we can establish a dual relation in space by making a point correspond to the plane through the points of osculation of the three osculating planes through the point, and a plane to the point of intersection of the three planes which osculate the cubic at the points where it is met by the plane. In particular, to a point on the cubic corresponds the osculating plane at the point, and vice versa.

8 Principal normal and binormal Evidently there are an infinity of normals to a curve at a point. Two of these are of particular interest: the normal, which lies in the osculating plane at the point, called the *principal normal*, and the normal, which is perpendicular to this plane, called the *binormal*.

If the direction-cosines of the binormal be denoted by λ, μ, ν , we have from (35)

$$\lambda \mu \nu = (y'z'' - z'y'')(z'x'' - x'z'')(x'y'' - y'x'')$$

In consequence of the identity

$$\Sigma(y'z'' - z'y'')^2 = \Sigma x'^2 \Sigma x''^2 - (\Sigma x'x'')^2,$$

the value of the common ratio is reducible by means of (19) and (28) to $\pm \rho$ *. We take the positive direction of the binormal to be such that this ratio shall be $+\rho$, then

$$(37) \quad \lambda = \rho(y'z'' - z'y''), \quad \mu = \rho(z'x'' - x'z''), \quad \nu = \rho(x'y'' - y'x'')$$

When the parameter u is general, these formulas are

$$(38) \quad \lambda = \frac{\rho}{\phi'^3} (f_2' f_3'' - f_3' f_2''), \quad \mu = \frac{\rho}{\phi'^3} (f_3' f_1'' - f_1' f_3''), \quad \nu = \frac{\rho}{\phi'^3} (f_1' f_2'' - f_2' f_1''),$$

or in other form

$$(38') \quad \lambda = \rho \frac{dy d^2 z - dz d^2 y}{ds^3}, \quad \mu = \rho \frac{dz d^2 x - dx d^2 z}{ds^3}, \quad \nu = \rho \frac{dx d^2 y - dy d^2 x}{ds^3}$$

* For $\Sigma x'x'' = 0$ as is seen by differentiating $\Sigma x'^2 = 1$ with respect to s

definition the principal normal is perpendicular to both the tangent and binormal. We make the convention that its positive direction is such that the positive directions of the tangent, principal normal and binormal at a point have the same mutual orientation as the positive directions

x -, y -, z -axes respectively. The directions are represented in Fig. 2 by the lines MT , MC , MB . If l , m , n , denote the direction cosines of the principal normal,

*

$$\begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix} = +1,$$

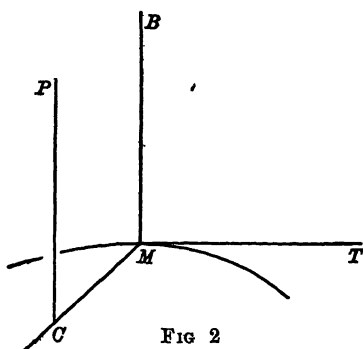


FIG 2

such it follows that

$$\begin{cases} \alpha = m\nu - n\mu, & \beta = n\lambda - l\nu, & \gamma = l\mu - m\lambda, \\ l = \mu\gamma - \nu\beta, & m = \nu\alpha - \lambda\gamma, & n = \lambda\beta - \mu\alpha, \\ \lambda = \beta n - \gamma m, & \mu = \gamma l - \alpha n, & \nu = \alpha m - \beta l \end{cases}$$

Substituting the values of α , β , γ , λ , μ , ν from (19) and (37) in the equations for l , m , n , the resulting equations are reducible to

$$l = \rho x'', \quad m = \rho y'', \quad n = \rho z''$$

when the parameter u is general, we have

$$\frac{\rho}{\phi'^3} (\phi' f_1'' - \phi'' f_1'), \quad m = \frac{\rho}{\phi'^3} (\phi' f_2'' - \phi'' f_2'), \quad n = \frac{\rho}{\phi'^3} (\phi' f_3'' - \phi'' f_3'),$$

or, in form,

$$\frac{ds d^2x - dx d^2s}{ds^3}, \quad m = \frac{ds d^2y - dy d^2s}{ds^3}, \quad n = \frac{ds d^2z - dz d^2s}{ds^3}$$

In consequence of (29) equations (42) may be written

$$l = \rho \frac{d\alpha}{ds}, \quad m = \rho \frac{d\beta}{ds}, \quad n = \rho \frac{d\gamma}{ds},$$

in terms of (27),

$$l = \frac{d\alpha}{d\sigma}, \quad m = \frac{d\beta}{d\sigma}, \quad n = \frac{d\gamma}{d\sigma}$$

The tangent to the spherical indicatrix of a curve is parallel to the principal normal to the curve and has the same sense

9. Osculating circle **Center of first curvature** We have defined the osculating plane to a curve at a point M to be the limiting position of the plane determined by the tangent at M and by a point M_1 of the curve, as the latter approaches M along the curve. We consider now the circle in this plane which has the same tangent at M as the curve, and passes through M_1 . The limiting position of this circle, as M_1 approaches M , is called the *osculating circle* to the curve at M . It is evident that its center C_0 is on the principal normal at M . Hence, with reference to any fixed axes in space, the coordinates of C_0 , denoted by X_0, Y_0, Z_0 , are of the form

$$X_0 = x + rl, \quad Y_0 = y + rm, \quad Z_0 = z + rn,$$

where the absolute value of r is the radius of the osculating circle.

In order to find the value of r , we return to the consideration of the circle, when M_1 does not have its limiting position, and we let X, Y, Z, l, m, n, r_1 denote respectively coordinates of the center of the circle, the direction cosines of the diameter through M and the radius. If x_1, y_1, z_1 be the coordinates of M_1 , they have the values (17), and since M_1 is on the circle, we have

$$r_1^2 = \Sigma(X - x_1)^2 = \Sigma(r_1 l_1 - ex' - \frac{1}{2} e^2 x'')^2$$

If we notice that $\Sigma x' l_1 = 0$, and after reducing the above equation divide through by e^2 , we have

$$1 - r_1 \Sigma l_1 x'' + \eta = 0,$$

where η involves terms of the first and higher orders in e . In the limit r_1 becomes r , $\Sigma x'' l_1$ becomes $\Sigma x'' l$, that is $\frac{1}{\rho}$, and this equation reduces to

$$1 - \frac{r}{\rho} = 0,$$

so that r is equal to the radius of curvature. On this account the osculating circle is called the *circle of curvature* and its center the *center of first curvature* for the point. Since r is positive the center of curvature is on the positive half of the principal normal, and consequently its coordinates are

$$(44) \quad X_0 = x + \rho l, \quad Y_0 = y + \rho m, \quad Z_0 = z + \rho n$$

he line normal to the osculating plane at the center of curvature is called the *polar line* or *polar* of the curve for the corresponding point. Its equations are

$$\frac{X-x-\rho l}{\lambda} = \frac{Y-y-\rho m}{\mu} = \frac{Z-z-\rho n}{\nu}$$

Polar

fig 2 C represents the center of curvature and CP the polar for M

curve may be looked upon as the path of a point moving under the action of a system of forces. From this point of view it is convenient to take for parameter time which has elapsed since the point passed a given position. Let t denote this parameter. As t is a function of s , we have

$$\frac{dx}{dt} = \frac{dx}{ds} \frac{ds}{dt} = \alpha \frac{ds}{dt}, \quad \frac{dy}{dt} = \beta \frac{ds}{dt}, \quad \frac{dz}{dt} = \gamma \frac{ds}{dt}$$

the rate of change of the position of the point with the time, or its *velocity*, may be represented by the length $\frac{ds}{dt}$ laid off on the tangent to the curve. In like manner, by means of (41), we have

$$\frac{d^2x}{dt^2} = \alpha \frac{d^2s}{dt^2} + \frac{l}{\rho} \left(\frac{ds}{dt} \right)^2, \quad \frac{d^2y}{dt^2} = \beta \frac{d^2s}{dt^2} + \frac{m}{\rho} \left(\frac{ds}{dt} \right)^2, \quad \frac{d^2z}{dt^2} = \gamma \frac{d^2s}{dt^2} + \frac{n}{\rho} \left(\frac{ds}{dt} \right)^2$$

this it is seen that the rate of change of the velocity at a point, or the *acceleration*, may be represented by a vector in the osculating plane at the point, perpendicular to the tangent and whose components on the tangent and principal normal are $\frac{ds}{dt}$ and $\frac{1}{\rho} \left(\frac{ds}{dt} \right)^2$

EXAMPLES

Prove that the curvature of a plane curve defined by the equation $M(x, y)dx + N(x, y)dy = 0$ is

$$\frac{1}{\rho} = \frac{MN \left(\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} \right) - N^2 \frac{\partial M}{\partial x} - M^2 \frac{\partial N}{\partial y}}{(M' + N')^{\frac{3}{2}}}$$

Show that the normal planes to the curve,

$$x = a \sin^2 u, \quad y = a \sin u \cos u, \quad z = a \cos u,$$

pass through the origin, and find the spherical indicatrix of the curve.

The straight line is the only real curve of zero curvature at every point.

Derive the following properties of the twisted cubic.

1. In any plane there is one line, and only one, through which two osculating planes can be drawn.

2. Four fixed osculating planes are cut by the line of intersection of any two osculating planes in four points whose cross-ratio is constant.

3. Four planes through a variable tangent and four fixed points of the curve have a constant cross-ratio.

4. What is the dual of (c) by the results of § 7?

5 Determine the form of the function ϕ so that the principal normals to the curve $x = u$, $y = \sin u$, $z = \phi(u)$ are parallel to the yz plane

6 Find the osculating plane and radius of first curvature of

$$x = a \cos u + b \sin u, \quad y = a \sin u + b \cos u, \quad z = c \sin 2u$$

10 Torsion Frenet-Serret formulas It has been seen that, unless a curve be plane, the osculating plane varies as the point moves along the curve. The change in the direction depends evidently upon the form of the curve. The ratio of the angle $\Delta\theta_1$ between the binormals at two points of the curve and their curvilinear distance Δs expresses our idea of the mean change in the direction of the osculating plane. And so we take the limit of this ratio, as one point approaches the other, as the measure of the rate of this change at the latter point. This limit is called the *second curvature*, or *torsion*, of the curve, and its inverse the *radius of second curvature*, or the *radius of torsion*. The latter will be denoted by τ .

In order to establish the existence of this limit and to find an expression for it in terms of the functions defining the curve, we draw radii of the unit sphere parallel to the positive binormals of the curve and take the locus of the end points of these radii as a second spherical representation of the curve. The coordinates of points of this representative curve on the sphere are λ , μ , ν . Proceeding in a manner similar to that in § 6, we obtain the equation

$$(46) \quad \frac{1}{\tau^2} = \frac{d\sigma_1^2}{ds^2} = \left(\frac{d\lambda}{ds}\right)^2 + \left(\frac{d\mu}{ds}\right)^2 + \left(\frac{d\nu}{ds}\right)^2,$$

where $d\sigma_1$ is the linear element of the spherical indicatrix of the binormals.

In order that a real curve have zero torsion at every point, the cosines λ , μ , ν must be constant. By a change of the fixed axes, which evidently has no effect upon the form of the curve, the cosines can be given the values $\lambda = 1$, $\mu = \nu = 0$. It follows from (40) that $\alpha = 0$, and consequently $x = \text{const}$. Hence a necessary and sufficient condition that the torsion of a real curve be zero at every point is that the curve be plane.

In the subsequent discussion we shall need the derivatives with respect to s of the direction-cosines α , β , γ , l , m , n , λ , μ , ν . We deduce them now. From (41) we have

$$(47) \quad \alpha' = \frac{l}{\rho}, \quad \beta' = \frac{m}{\rho}, \quad \gamma' = \frac{n}{\rho}$$

der to find the values of λ' , μ' , ν' , we differentiate with respect to s the identities,

$$\lambda^2 + \mu^2 + \nu^2 = 1, \quad \alpha\lambda + \beta\mu + \gamma\nu = 0,$$

consequence of (47), obtain

$$\lambda\lambda' + \mu\mu' + \nu\nu' = 0, \quad \alpha\lambda' + \beta\mu' + \gamma\nu' = 0$$

hence, by (40), follows the proportion

$$\lambda' : \mu' : \nu' = l : m : n,$$

the factor of proportionality is $\pm 1/\tau$, as is seen from (46). The algebraic sign of τ is not determined by the latter equation. Its sign is given by writing the above proportion thus

$$\lambda' = \frac{l}{\tau}, \quad \mu' = \frac{m}{\tau}, \quad \nu' = \frac{n}{\tau}$$

The identity $l = \mu\gamma - \nu\beta$ be differentiated with respect to s . It is reducible by (40), (47), and (48) to

$$l' = -\left(\frac{\alpha}{\rho} + \frac{\lambda}{\tau}\right)$$

Similar expressions can be found for m' and n' . Gathering together these results, we have the following formulas fundamental in the theory of twisted curves, and called the Frenet-Serret formulas

$$\begin{cases} \alpha' = \frac{l}{\rho}, & \beta' = \frac{m}{\rho}, & \gamma' = \frac{n}{\rho}, \\ l' = -\left(\frac{\alpha}{\rho} + \frac{\lambda}{\tau}\right), & m' = -\left(\frac{\beta}{\rho} + \frac{\mu}{\tau}\right), & n' = -\left(\frac{\gamma}{\rho} + \frac{\nu}{\tau}\right), \\ \lambda' = \frac{l}{\tau}, & \mu' = \frac{m}{\tau}, & \nu' = \frac{n}{\tau} \end{cases}$$

For example, we derive another expression for the torsion equation

$$\lambda = \rho(y'z'' - z'y'')$$

differentiated with respect to s , the result may be written

$$\frac{l}{\tau} = \frac{\rho'}{\rho}\lambda + \rho(y'z''' - z'y''')$$

The equation and similar ones for m/τ , n/τ be multiplied by l , m , n respectively and added, we have, in consequence of (50) and (41),

$$\frac{1}{\tau} = -\rho' \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}$$

The last three of equations (50) give the rate of change of the direction-cosines of the osculating plane of a curve as the point of osculation moves along the curve. From these equations it follows that a necessary and sufficient condition that this rate of change at a point be zero is that the values of s for the point make the determinant in equation (51) vanish. At such a point the osculating-plane is said to be *stationary*.

11 **Form of curve in the neighborhood of a point** **The sign of torsion** We have made the convention that the positive directions of the tangent, principal normal, and binormal shall have the same relative orientation as the fixed x, y, z axes respectively. When we take these lines at a point M_0 for axes, the equations of the curve can be put in a very convenient form. If the coordinates be expressed in terms of the arc measured from M_0 , we have from (19) and (41) that for $s = 0$

$$x' = 1, \quad y' = z' = 0, \quad x'' = 0, \quad y'' = \frac{1}{\rho}, \quad z'' = 0$$

When the values of l and λ from (41) and (37) are substituted in the fourth of equations (50), we obtain

$$(52) \quad x''' = -\frac{x'}{\rho^2} - \frac{1}{\tau} (y'z'' - z'y'') - \frac{\rho'}{\rho} x''$$

From this and similar expressions for y''' and z''' we find that for $s = 0$

$$x''' = -\frac{1}{\rho^2}, \quad y''' = -\frac{\rho'}{\rho^2}, \quad z''' = -\frac{1}{\rho\tau}$$

Hence, by Maclaurin's theorem, the coordinates x, y, z can be expressed in the form

$$(53) \quad \begin{cases} x = s - \frac{1}{6\rho^2}s^3 + \dots, \\ y = \frac{s^2}{2\rho} - \frac{1}{6}\frac{\rho'}{\rho^2}s^3 + \dots, \\ z = -\frac{1}{6\rho\tau}s^3 + \dots, \end{cases}$$

where ρ and τ are the radii of first and second curvature at the point $s = 0$, and the unwritten terms are of the fourth and higher powers in s .

From the last of these equations it is seen that for sufficiently small values of s the sign of z changes with the sign of s unless

at M_0 . Hence, unless the osculating plane is stationary at the curve crosses the plane at the point *. Furthermore, point moves along a curve in the positive direction, it from the positive to the negative side of the osculating plane at a point, or vice-versa, according as the torsion at the point is positive or negative. In the former case the curve is *sinistrorsum*, in the latter *dextrorsum*.

Another consequence of this equation, we remark that as a point M on the curve approaches M_0 , the distance from M to the osculating plane at M_0 is of the third order of magnitude in comparison with MM_0 . By means of the other equations (53) we find that the distance to any other plane through M_0 is of the same order at most. Hence we have the theorem

The distance from a point on a twisted curve to its osculating plane at an ordinary point is of the third order of magnitude in comparison with the distance from the point to the curve, and of all the planes through the point it lies closest to the curve.

In the second of (53) it is seen that y is positive for sufficiently small values of s , positive or negative. Hence, in the neighborhood of an ordinary point, the curve lies entirely on one side of the plane determined by the tangent and binormal — on the side of the positive direction of the principal normal.

Other properties of a twisted curve are discovered, likewise, by consideration of the projections upon the coordinate planes. The curve is approximated by a curve, whose equations consist of the first two of (53). The projection on the osculating plane is the curve $x = s$, $y = s^2/2\rho$, whose axis is the principal normal. On the plane of the tangent and binormal it is the curve $x = s$, $z = -s^3/6\rho\tau$, which has the tangent to the curve as an inflectional tangent. And the curve projects upon the plane of the binormal and principal normal into the semiparabola $y = s^2/2\rho$, $z = -s^3/6\rho\tau$, with the latter for the axis of the parabola.

The results are represented by the following figures, which picture the projections of the curve upon the osculating plane, normal plane, and the plane of the tangent and binormal. In the third figure the heavy line corresponds to the case where τ is positive and the dotted line to the case where τ is negative.

* This result can be derived readily by geometrical considerations.

The preceding results serve also to give a means of determining the variation in the osculating plane as the point moves along the curve. By means of (50) the direction cosines λ , μ , ν can be given the form

$$\lambda = \lambda_0 + \frac{l_0}{\tau_0}s + \dots, \quad \mu = \mu_0 + \frac{m_0}{\tau_0}s + \dots, \quad \nu = \nu_0 + \frac{n_0}{\tau_0}s + \dots,$$

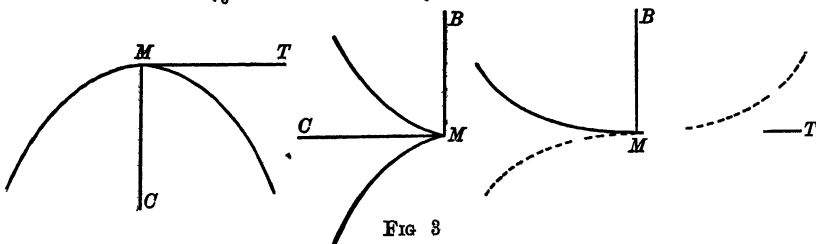


FIG. 3

where the subscript null indicates the value of a function for $s = 0$ and the un-written terms are of the second and higher terms in s . If the coordinate axes are those which lead to (53), the values of λ , μ , ν for the point of parameter δs are

$$\lambda = 0, \quad \mu = \frac{\delta s}{\tau_0}, \quad \nu = 1,$$

to within terms of higher order, and consequently the equation of this osculating plane at this point M_1 is

$$Y \frac{\delta s}{\tau_0} + Z = 0$$

If we put $Y = \rho_0$, we get the z -coordinate of the point in which this plane is cut by the polar line for the point $s = 0$, it is $-\rho_0 \delta s / \tau_0$. Hence, according as τ_0 is positive or negative at M , the osculating plane at the near-by point M_1 cuts the polar line for M on the negative or positive side of the osculating plane at M .

12 Cylindrical helices

As another example of the use of formulas (50) we derive several properties of cylindrical helices. By definition, a cylindrical helix is a curve which lies upon a cylinder and cuts the elements of the cylinder under constant angle. If the axis of z be taken parallel to the elements of the cylinder, we have $\gamma = \text{const}$. Hence, from (50),

$$\alpha = 0, \quad \frac{\gamma}{\rho} + \frac{\nu}{\tau} = 0, \quad \nu' = 0,$$

from which it follows that the cylindrical helices have the following properties

The principal normal is perpendicular to the element of the cylinder at the point, and consequently coincides with the normal to the cylinder at the point (§ 22)

The radii of first and second curvature are in constant ratio

Bertrand has established the converse theorem. *Every curve whose radii of first and second curvature are in constant ratio is a cylindrical helix.* In order to prove it, we put $\tau = \kappa \rho$, and remark from (50) that

$$\frac{d\alpha}{ds} = \kappa \frac{d\lambda}{ds}, \quad \frac{d\beta}{ds} = \kappa \frac{d\mu}{ds}, \quad \frac{d\gamma}{ds} = \kappa \frac{d\nu}{ds},$$

from which we get $\alpha = \kappa \lambda + a$, $\beta = \kappa \mu + b$, $\gamma = \kappa \nu + c$,

where a , b , c are constants. From these equations we find

$$a^2 + b^2 + c^2 = 1 + \kappa^2, \quad a\alpha + b\beta + c\gamma = 1$$

the tangents to the curve make the constant angle $\cos^{-1} \frac{1}{\sqrt{1+\kappa^2}}$ with the direction-cosines are $\frac{a, b, c}{\sqrt{1+\kappa^2}}$. Consequently the curve is a cylindrical helix and the elements of the helix have the above direction.

EXAMPLES

Find the length of the curve $x = a(u - \sin u)$, $y = a \cos u$, between the points u has the values $-\pi$ and π , show that the locus of the center of curvature has the same form as the given curve

Find the coordinates of the center of curvature of

$$x = a \cos u, \quad y = a \sin u, \quad z = a \cos 2u$$

Find the radius of curvature and torsion of

$$x = a(u - \sin u), \quad y = a(1 - \cos u), \quad z = bu$$

the principal normals of a curve are parallel to a fixed plane, the curve is a cylindrical helix

Show that the curve $x = e^u$, $y = e^{-u}$, $z = \sqrt{2}u$ is a cylindrical helix and that the section of the cylinder is a catenary, also that the curve lies upon a cylindrical surface whose right section is an equilateral hyperbola. Express the coordinates in terms of arc length and find the radius of first and second curvature

Show that if θ and ϕ denote the angles which the tangent and binormal to a curve make with a fixed line in space, then $\frac{\sin \theta d\theta}{\sin \phi d\phi} = \frac{\tau}{\rho}$

When two curves are symmetric with respect to the origin, their radii of curvature are equal and their radii of torsion differ only in sign

The osculating circle at an ordinary point of a curve has contact of the second order with the latter, and all other circles which lie in the osculating plane and are tangent to the curve at the point have contact of the first order

A necessary and sufficient condition that the osculating circle at a point have contact of the third order is that $\rho' = 0$ and $1/\tau = 0$ at the point, at such a point is said to *superosculate* the curve

Show that any twisted curve may be defined by equations of the form

$$\begin{aligned} x &= s - \frac{1}{6\rho^2}s^3 + \frac{1}{8}\frac{\rho}{\rho^3}s^4 + \dots, \\ y &= \frac{s^2}{2\rho} - \frac{1}{6}\frac{\rho}{\rho^2}s^3 + \frac{1}{24}\left[\left(\frac{1}{\rho}\right)' - \frac{1}{\rho^3} - \frac{1}{\rho\tau^2}\right]s^4 + \dots, \\ z &= -\frac{1}{6\rho\tau}s^3 - \frac{1}{24}\left[\left(\frac{1}{\rho\tau}\right)' + \frac{1}{\tau}\left(\frac{1}{\rho}\right)''\right]s^4 + \dots, \end{aligned}$$

and τ are the radii of first and second curvature at the point $s = 0$

When the equations of a curve are in the form (4), the torsion is given by

$$\frac{1}{\tau} = -\frac{\rho^2}{\phi'^6} \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix},$$

as the significance of equation (12)

12 The locus of the centers of curvature of a twisted curve of constant curvature is a curve of the same kind

13 When all the osculating planes of a curve pass through a fixed point curve is plane

14 Determine $f(u)$ so that the curve $x = a \cos u$, $y = a \sin u$, $z = f(u)$ shall be a plane curve. What is the form of the curve?

13 Intrinsic equations Fundamental theorem Let C_1 and C_2 be two curves defined in terms of their respective arcs s , and let points upon each with the same values of s correspond. We assume furthermore, that at corresponding points the radii of first curvature have the same value, and also the radii of second curvature. We shall show that C_1 and C_2 are congruent.

By a motion in space the points of the two curves for which $s = 0$ can be made to coincide in such a way that the tangents, principal normals, and binormals to them at the point coincide also. Hence if we use the notation of the preceding sections to indicate by subscripts 1 and 2 the functions of C_1 and C_2 , we have when $s = 0$,

$$(54) \quad x_1 = x_2, \quad \alpha_1 = \alpha_2, \quad l_1 = l_2, \quad \lambda_1 = \lambda_2,$$

and other similar equations

The Frenet-Serret formulas for the two curves are

$$\begin{aligned} \frac{d\alpha_1}{ds} &= \frac{l_1}{\rho}, & \frac{dl_1}{ds} &= -\left(\frac{\alpha_1}{\rho} + \frac{\lambda_1}{\tau}\right), & \frac{d\lambda_1}{ds} &= \frac{l_1}{\tau}, \\ \frac{d\alpha_2}{ds} &= \frac{l_2}{\rho}, & \frac{dl_2}{ds} &= -\left(\frac{\alpha_2}{\rho} + \frac{\lambda_2}{\tau}\right), & \frac{d\lambda_2}{ds} &= \frac{l_2}{\tau}, \end{aligned}$$

the functions without subscripts being the same for both curves. If the equations of the first row be multiplied by α_2 , l_2 , λ_2 respectively, and of the second row by α_1 , l_1 , λ_1 , and all added, we have

$$(55) \quad \frac{d}{ds} (\alpha_1 \alpha_2 + l_1 l_2 + \lambda_1 \lambda_2) = 0,$$

and consequently $\alpha_1 \alpha_2 + l_1 l_2 + \lambda_1 \lambda_2 = \text{const}$

This constant is equal to unity for $s = 0$, as is seen from (54); hence for all values of s we have

$$\alpha_1 \alpha_2 + l_1 l_2 + \lambda_1 \lambda_2 = 1$$

Using this equation with the identities

$$\alpha_1^2 + l_1^2 + \lambda_1^2 = 1, \quad \alpha_2^2 + l_2^2 + \lambda_2^2 = 1,$$

$$\text{in} \quad (\alpha_1 - \alpha_2)^2 + (l_1 - l_2)^2 + (\lambda_1 - \lambda_2)^2 = 0$$

so $\alpha_1 = \alpha_2$, $l_1 = l_2$, $\lambda_1 = \lambda_2$. Moreover, since in like manner $\gamma_1 = \gamma_2$, we have

$$\frac{d}{ds}(x_1 - x_2) = 0, \quad \frac{d}{ds}(y_1 - y_2) = 0, \quad \frac{d}{ds}(z_1 - z_2) = 0$$

hence frequently the differences $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$ are constant. For $s = 0$ they are zero, and so we have the theorem

Curves whose radii of first and second curvature are the same at corresponding points of the arc are congruent

From this it follows that a curve is determined, to within its position in space, by the expressions for the radii of first and second curvature in terms of the arc. And so the equations of a curve can be written in the form

$$\rho = f_1(s), \quad \tau = f_2(s)$$

These are called its *intrinsic equations*.

We now inquire, conversely, whether two equations (56), in which f_1 and f_2 are any functions whatever of a parameter s , are intrinsic equations of a curve for which s is the length of arc. In answering this question we show, in the first place, that the

$$\frac{du}{ds} = \frac{v}{\rho}, \quad \frac{dv}{ds} = -\left(\frac{u}{\rho} + \frac{w}{\tau}\right), \quad \frac{dw}{ds} = \frac{v}{\tau}$$

admit three sets of solutions, namely

$$\alpha, v = l, w = \lambda, u = \beta, v = m, w = \mu, u = \gamma, v = n, w = \nu,$$

where $\alpha, \beta, \gamma, \lambda, \mu, \nu$ are such that for each value of s the quantities $\alpha, \beta, \gamma, \lambda, \mu, \nu$ are the direction cosines of three mutually perpendicular lines. In fact, we know* that a system (57) admits of a set of solutions whose values for $s = 0$ are given arbitrarily. Consequently these equations admit of three sets of solutions

whose values for $s = 0$ are 1, 0, 0, 0, 1, 0, 0, 0, 1 respectively. By an argument similar to that applied to equation (55) we prove that for all values of s the solutions (58) satisfy the condition

$$(59) \quad \alpha\beta + lm + \lambda\mu = 0, \quad \beta\gamma + mn + \mu\nu = 0, \quad \gamma\alpha + nl + \nu\lambda = 0$$

In like manner, since it follows from (57) that

$$u \frac{du}{ds} + v \frac{dv}{ds} + w \frac{dw}{ds} = 0,$$

we prove that these solutions satisfy the conditions

$$(60) \quad \alpha^2 + l^2 + \lambda^2 = 1, \quad \beta^2 + m^2 + \mu^2 = 1, \quad \gamma^2 + n^2 + \nu^2 = 1$$

But the conditions (59), (60) are equivalent to (40), and consequently the three sets of functions $\alpha, \beta, \gamma, l, m, n, \lambda, \mu, \nu$ are the direction-cosines of three mutually perpendicular lines for all values of s .

Suppose we have such a set of solutions. For the curve

$$(61) \quad x = \int \alpha ds, \quad y = \int \beta ds, \quad z = \int \gamma ds,$$

the functions α, β, γ are the direction-cosines of the tangent, since $ds^2 = dx^2 + dy^2 + dz^2$, s measures the arc of the curve. In (61) and the first of (57) we get

$$\frac{d^2x}{ds^2} = \frac{l}{\rho}, \quad \frac{d^2y}{ds^2} = \frac{m}{\rho}, \quad \frac{d^2z}{ds^2} = \frac{n}{\rho}, \quad \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 = \frac{1}{\rho^2}$$

Hence if ρ be positive for all values of s , it is the radius of curvature of the curve (61), and l, m, n are the direction-cosines of the principal normal in the positive sense. In consequence of (40) the functions λ, μ, ν are the direction-cosines of the binormal, and from (50) and the third of (57) it follows that τ is the radius of torsion of the curve. Therefore we have the following theorems fundamental in the theory of curves.

Given any two analytic functions, $f_1(s), f_2(s)$, of which the first is positive for all values of s within a certain domain, there exists a curve for which $\rho = f_1(s), \tau = f_2(s)$, and s is the arc, for values of s in the given domain. The determination of the curve reduces to the determination of three sets of solutions of equations (57), satisfying the conditions (59), (60), and to quadratures.

We proceed now to the integration of equations (57). Since each of integrals of the desired kind must satisfy the relation

$$u^2 + v^2 + w^2 = 1,$$

we introduce with Darboux* two functions σ and ω , defined by

$$\begin{cases} \frac{u + iv}{1 - w} = \frac{1 + w}{u - iv} = \sigma, \\ \frac{u - iv}{1 - w} = \frac{1 + w}{u + iv} = -\frac{1}{\omega} \end{cases}$$

is evident that the functions σ and $-\frac{1}{\omega}$ are conjugate imaginaries. Solving for u, v, w , we get

$$u = \frac{1 - \sigma\omega}{\sigma - \omega}, \quad v = i \frac{1 + \sigma\omega}{\sigma - \omega}, \quad w = \frac{\sigma + \omega}{\sigma - \omega}$$

these values be substituted in equations (57), it is found that the functions σ and ω are solutions of the equation

$$\frac{d\theta}{ds} = \frac{i}{2\tau} - \frac{i}{\rho}\theta - \frac{i}{2\tau}\theta^2$$

and conversely, any two different solutions of (65), when substituted in (64), lead to a set of solutions of equations (57) satisfying the relation (62). Our problem reduces then to the integration of equation (65).

14 Riccati equations Equation (65) may be written

$$\frac{d\theta}{ds} = L + 2M\theta + N\theta^2,$$

where L, M, N are functions of s . This equation is a generalized form of an equation first studied by Riccati,† and consequently named for him. As Riccati equations occur frequently in the theory of curves and surfaces, we shall establish several of their properties.

Theorem *When a particular integral of a Riccati equation is known, the general integral can be obtained by two quadratures*

* *Leçons sur la Théorie Générale des Surfaces* Vol I, p 22. We shall refer to this text frequently, and for brevity give our references the form Darboux I 22.

† Cf. Forsyth, *Differential Equations*, chap v also Cohen, *Differential Equations*, 173-177.

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particular integral of (66) If we put $\theta = 1/\phi + \theta_1$,
for the determination of ϕ is

$$\frac{d\phi}{ds} + 2(M + N\theta_1)\phi + N = 0$$

equation is linear and of the first order, it can be solved by two quadratures Since the general integral of (67) is of the form $\phi = f_1(s) + af_2(s)$, where a denotes the constant of integration, the general integral of equation (66) is of the form

$$(68) \quad \theta = \frac{aP + Q}{aR + S},$$

where P, Q, R, S are functions of s

Theorem *When two particular integrals of a Riccati equation are known, the general integral can be found by one quadrature*

Let θ_1 and θ_2 be two solutions of equation (66) If we effect the substitution $\theta = \frac{1}{\psi} + \theta_2$, the equation in ψ is

$$\frac{d\psi}{ds} + 2(M + N\theta_2)\psi + N = 0$$

If this equation and (67) be multiplied by $1/\psi$ and $1/\phi$ respectively, and subtracted, the resulting equation is reducible to $\frac{d}{ds}(\psi/\phi) = N(\theta_1 - \theta_2)\psi/\phi$ Consequently the general integral of (66) is given by

$$(69) \quad \frac{\theta - \theta_1}{\theta - \theta_2} = \frac{\psi}{\phi} = ae^{\int N(\theta_1 - \theta_2) ds},$$

where a is the constant of integration

Since equation (68) may be looked upon as a linear fractional substitution upon a , four particular solutions $\theta_1, \theta, \theta_3, \theta_4$, corresponding to four values a_1, a_2, a_3, a_4 of a , are in the same cross-ratio as these constants Hence we have the theorem

The cross-ratio of any four particular integrals of a Riccati equation is constant

From this it follows that if three particular integrals are known, the general integral can be obtained without quadrature

the determination of the coordinates of a curve defined by the equations (56). We return to the consideration of equation (55) and indicate by

$$\sigma_i = \frac{a_i P + Q}{a_i R + S}, \quad \omega_i = \frac{b_i P + Q}{b_i R + S}, \quad (i=1, 2, 3)$$

the particular integrals of this equation. From these we obtain a set of solutions of equations (57), namely

$$\alpha = \frac{1 - \sigma_1 \omega_1}{\sigma_1 - \omega_1}, \quad \beta = \frac{1 + \sigma_1 \omega_1}{\sigma_1 - \omega_1}, \quad \gamma = \frac{\sigma_1 + \omega_1}{\sigma_1 - \omega_1},$$

and similar expressions in $\sigma_2, \omega_2, \sigma_3, \omega_3$ respectively for β, m, μ . These expressions satisfy the conditions (60). In order that the conditions (60) also may be satisfied we must have

$$\frac{\sigma_i - \sigma_k}{\sigma_i - \omega_k} = \frac{\omega_i - \omega_k}{\omega_i - \omega_k} = -1,$$

which is reducible to

$$\frac{a_i - a_k}{a_i - b_k} = \frac{b_i - a_k}{b_i - b_k} = -1 \quad \left(\begin{matrix} i=1, 2, 3 \\ k=1, 2, 3 \quad i \neq k \end{matrix} \right)$$

which shows that two of the three pairs of constants $a_1, b_1, a_2, b_2, a_3, b_3$ are in harmonic range.

When the values (70) for σ_i, ω_i are substituted in the expressions (71), it is found that

$$\left\{ \begin{aligned} \alpha &= \frac{1 - a_1 b_1}{a_1 - b_1} U + \frac{1 + a_1 b_1}{a_1 - b_1} V + \frac{a_1 + b_1}{a_1 - b_1} W, \\ \beta &= \frac{1 - a_2 b_2}{a_2 - b_2} U + \frac{1 + a_2 b_2}{a_2 - b_2} V + \frac{a_2 + b_2}{a_2 - b_2} W, \\ \gamma &= \frac{1 - a_3 b_3}{a_3 - b_3} U + \frac{1 + a_3 b_3}{a_3 - b_3} V + \frac{a_3 + b_3}{a_3 - b_3} W, \end{aligned} \right.$$

for the sake of brevity, we have put

$$\left\{ \begin{aligned} U &= \frac{(P^2 - R^2) - (Q^2 - S^2)}{2(PS - QR)}, \\ V &= \frac{(P^2 - R^2) + (Q^2 - S^2)}{2(PS - QR)}, \\ W &= \frac{RS - PQ}{PS - QR} \end{aligned} \right.$$

The coefficients of U , V , and W in (73) are of the same form as the expressions (71) for α , l , λ , β , m , μ , γ , n , ν . Moreover, the equations of condition (59) are equivalent to (72). Hence these coefficients are the direction-cosines of three fixed directions in space mutually perpendicular to one another. If lines through the origin of coordinates parallel to these three lines be taken for a new set of axes, the expressions for α , β , γ with reference to these axes reduce to U , V , W respectively*. These results may be stated thus

If the general solution of equation (65) be

$$(68) \quad \theta = \frac{aP + Q}{aR + S},$$

the curve whose radii of first and second curvature are ρ and τ respectively is given by

$$x = \int \frac{(P^2 - R^2) - (Q^2 - S^2)}{2(PS - QR)} ds, \quad y = \int \frac{(P^2 - R^2) + (Q^2 - S^2)}{2(PS - QR)} ds, \\ z = \int \frac{RS - PQ}{PS - QR} ds \dagger$$

It must be remarked that the new axes of coordinates are not necessarily real, so that when it is important to know whether the curves are real it will be advisable to consider the general formulas (73). An example of this will be given later.

We shall apply the preceding results to several problems.

When the curve is plane the torsion is zero, and conversely. For this case equation (65) reduces to $\frac{d\theta}{ds} = -\frac{1}{\rho}\theta$, of which the general integral is

$$\theta = ae^{-\int \frac{ds}{\rho}} = ae^{-i\sigma},$$

where a is an arbitrary constant, and by (27) σ is the measure of the arc of the spherical indicatrix of the tangent. This solution is of the form (68), with

$$P = e^{-i\sigma}, \quad Q = R = 0, \quad S = 1$$

Therefore the coordinates are given by

$$(75) \quad x = \int \cos \sigma ds, \quad y = \int \sin \sigma ds, \quad z = 0$$

Hence the coordinates of any plane curve can be put in this form

* This is the same thing as taking

$$a_1 = -b_1 = 1, \quad a_2 = -b_2 = i, \quad a_3 = \infty, \quad b_3 = 0$$

† Scheffers, *Anwendung der Differential und Integral Rechnung auf Geometrie*, Vol I, p 219. Leipzig, 1902

is seen that *cylindrical helices* are characterized by the property that the first and second curvature are in constant ratio. If we put $\tau = \rho c$, equation (65) can be written

$$\frac{d\theta}{ds} = \frac{i}{2\tau} (1 - 2c\theta - \theta^2)$$

whose quadratic integrals are the roots of the equation $\theta^2 + 2c\theta - 1 = 0$. These roots are real and unequal if c is real, we consider only this case, and put

$$\theta_1 = -c - \sqrt{c^2 + 1}, \quad \theta_2 = -c + \sqrt{c^2 + 1}, \quad \theta_1 \theta_2 = -1$$

it follows that the general solution of the above equation is

$$\theta = \frac{ae^{it}\theta_2 - \theta_1}{ae^{it} - 1},$$

we have put

$$t = \frac{\theta_2 - \theta_1}{2} \int \frac{ds}{\tau} = \frac{\sqrt{c^2 + 1}}{c} \int \frac{ds}{\rho}$$

and $-\frac{1}{\omega}$ in (68) are conjugate imaginary, if we take

$$\sigma = \frac{ae^{it}\theta_2 - \theta_1}{ae^{it} - 1}, \quad \omega = \frac{be^{it}\theta_2 - \theta_1}{be^{it} - 1},$$

b must be such that

$$\frac{ae^{it}\theta_2 - \theta_1}{ae^{it} - 1} = -\frac{b_0 e^{-it} - 1}{b_0 e^{-it}\theta_2 - \theta_1},$$

b_0 denotes the conjugate imaginary of b . This reduces, in consequence of

$$ab_0 = -\frac{(1 + \theta_1^2)}{(1 + \theta_2^2)} = -\theta_1^2$$

the solution of this is given by taking ∞ and 0 for a and b , we put $a_i = \infty$, these values being substituted in (72), we get $a_i + b_i = 0$, where $i = 1, 2$. So equation (79) becomes $b_i b_{i0} = \theta_i^2$, where $i = 1, 2$. The solutions of this equation are $b_2 = -i\theta_1$. From (77) $P = e^{it}\theta_2$, $Q = -\theta_1$, $R = e^{it}$, $S = -1$, so that

$$\frac{c}{\sqrt{c^2 + 1}} (\theta_2 e^{it} - \theta_1 e^{-it}), \quad V = \frac{ic}{2\sqrt{c^2 + 1}} (\theta_2 e^{it} + \theta_1 e^{-it}), \quad W = \frac{1}{\sqrt{c^2 + 1}}$$

the foregoing values are substituted in (73), and the resulting values of (61), we get

$$x = \frac{c}{\sqrt{c^2 + 1}} \int \cos t ds, \quad y = \frac{c}{\sqrt{c^2 + 1}} \int \sin t ds, \quad z = \frac{s}{\sqrt{c^2 + 1}}$$

In the last of these expressions we find that the tangent to the curve makes an angle with the z -axis — the direction of the elements of the cylinder — whose cross-section of the cylinder is defined by

$$x_1 = \int \cos t ds_1, \quad y_1 = \int \sin t ds_1,$$

s_1 denotes the arc of this section measured from a point of it. If ρ_1 denotes the curvature of the right section, we find that $\rho c^2 = \rho_1(c^2 + 1)$

EXAMPLES

1 Find the coordinates of the cylindrical helix whose intrinsic equations are $\rho = \tau = s$

2 Show that the helix whose intrinsic equations are $\rho = \tau = (s^2 + 4)/\sqrt{2}$ lies upon a cylinder whose cross-section is a catenary

3 Establish the following properties for the curve with the intrinsic equations $\rho = as, \tau = bs$, where a and b are constants

(a) the Cartesian coordinates are reducible to $x = Ae^{ht} \cos t, y = Ae^{ht} \sin t, z = Be^{ht}$, where A, B, h are functions of a and b ,

(b) the curve lies upon a circular cone whose axis coincides with the z -axis and cuts the elements of the cone under constant angle

16 Moving trihedral In § 11 we took for fixed axes of reference the tangent, principal normal, and binormal to a curve at a point M_0 of it, and expressed the coordinates of any other point of the curve with respect to these axes as power series in the arc s of the curve between the two points. Since M_0 is any point of the curve, there is a set of such axes for each of its points. Hence, instead of considering only the points whose locus is the curve, we may look upon the moving point as the intersection of three mutually perpendicular lines which move along with the point, the whole figure rotating so that in each position the lines coincide with the tangent, principal normal, and binormal at the point. We shall refer to such a configuration as the *moving trihedral*. In the solution of certain problems it is of advantage to refer the curve to this moving trihedral as axes. We proceed to the consideration of this idea.

With reference to the trihedral at a point M , the direction-cosines of the tangent, principal normal, and binormal at M have the values

$$\alpha = 1, \beta = \gamma = 0, \quad l = 0, m = 1, n = 0, \quad \lambda = \mu = 0, \nu = 1$$

As the trihedral begins to move, the rates of change of these functions with s are found from the Frenet formulas (50) to have the values

$$\begin{aligned} \frac{d\alpha}{ds} = 0, \quad \frac{d\beta}{ds} = \frac{1}{\rho}, \quad \frac{d\gamma}{ds} = 0, \quad \frac{dl}{ds} = -\frac{1}{\rho}, \quad \frac{dm}{ds} = 0, \\ \frac{dn}{ds} = -\frac{1}{\tau}, \quad \frac{d\lambda}{ds} = 0, \quad \frac{d\mu}{ds} = \frac{1}{\tau}, \quad \frac{d\nu}{ds} = 0 \end{aligned}$$

η, ζ denote coordinates referring to the axes at M , and those with reference to the axes at M' , and let $MM' = \Delta s$. Since the rate of change of α is zero and $\alpha = 1$ at M , the cosine of the angle between the ξ - and ξ' -axes is 1 to within higher than the first order in Δs . Likewise the cosine of the angle between the ξ - and η' -axes is $-\Delta s/\rho$. We calculate the cosines of the angles between all the axes and the results may be expressed as follows

ξ	η	ζ
1	$\frac{\Delta s}{\rho}$	0
$-\frac{\Delta s}{\rho}$	1	$-\frac{\Delta s}{\tau}$
0	$\frac{\Delta s}{\tau}$	1

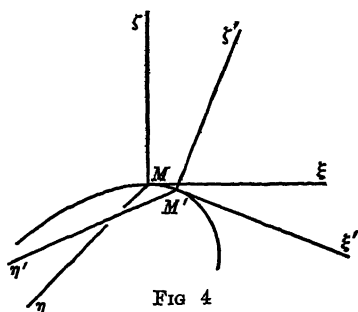


FIG. 4

be a point whose coordinates with respect to the trihedral ξ, η, ζ . Suppose that as M describes the given curve C , P describes a path Γ . It may happen that in this motion P is fixed to the moving trihedral, but in general the change in the position of P will be due not only to the motion of the trihedral but also to a motion relative to it. In the latter general case, if P' be the point on Γ corresponding to M' on C , the coordinates of P' relative to the axes at M and M' may be written

$$\Delta_1 \xi, \eta + \Delta_1 \eta, \zeta + \Delta_1 \zeta, \quad \xi + \Delta \xi, \eta + \Delta \eta, \zeta + \Delta \zeta$$

Δ indicates the variation of a function θ relative to the trihedral, and $\Delta_1 \theta$ the variation due to the latter and to the motion of the trihedral.

In terms of higher order the coordinates of M' are expressible with respect to the axes at M , and with the aid of (81) the coordinates of the transformation of coordinates with respect to the axes are expressible thus

$$\begin{aligned} + \Delta_1 \xi &= \Delta s + (\xi + \Delta \xi) - (\eta + \Delta \eta) \frac{\Delta s}{\rho}, \\ + \Delta_1 \eta &= (\xi + \Delta_2 \xi) \frac{\Delta s}{\rho} + (\eta + \Delta \eta) + (\zeta + \Delta \zeta) \frac{\Delta s}{\tau}, \\ + \Delta_1 \zeta &= -(\eta + \Delta \eta) \frac{\Delta s}{\tau} + (\zeta + \Delta \zeta) \end{aligned}$$

These reduce to

$$\begin{aligned}\frac{\Delta_1 \xi}{\Delta s} &= \frac{\Delta_2 \xi}{\Delta s} + 1 - \frac{\eta + \Delta_2 \eta}{\rho}, & \frac{\Delta_1 \zeta}{\Delta s} &= \frac{\Delta_2 \zeta}{\Delta s} - \frac{\eta + \Delta_2 \eta}{\tau}, \\ \frac{\Delta_1 \eta}{\Delta s} &= \frac{\Delta_2 \eta}{\Delta s} + \frac{\xi + \Delta_2 \xi}{\rho} + \frac{\zeta + \Delta_2 \zeta}{\tau}\end{aligned}$$

In the limit as M' approaches M these equations become

$$(82) \quad \frac{\delta \xi}{ds} = \frac{d\xi}{ds} + 1 - \frac{\eta}{\rho}, \quad \frac{\delta \eta}{ds} = \frac{d\eta}{ds} + \frac{\xi}{\rho} + \frac{\zeta}{\tau}, \quad \frac{\delta \zeta}{ds} = \frac{d\zeta}{ds} - \frac{\eta}{\tau},$$

thus $\frac{\delta \theta}{ds}$ denotes the absolute rate of change of θ , and $\frac{d\theta}{ds}$ the relative to the trihedral *

If t denotes the distance between P and a point P_1 (ξ_1, η_1, ζ_1) that is $t^2 = (\xi_1 - \xi)^2 + (\eta_1 - \eta)^2 + (\zeta_1 - \zeta)^2$, we find by means of the formulas (82) that

$$\frac{dt}{ds} = \frac{\delta t}{ds}$$

If a, b, c denote the direction-cosines of PP_1 with respect to the axes at M , then

$$\xi_1 = \xi + at, \quad \eta_1 = \eta + bt, \quad \zeta_1 = \zeta + ct$$

When we express the condition that ξ_1, η_1, ζ_1 as well as ξ, η , satisfy equations (82), we are brought to the following fundamental relations between the variations of a, b, c

$$(83) \quad \frac{\delta a}{ds} = \frac{da}{ds} - \frac{b}{\rho}, \quad \frac{\delta b}{ds} = \frac{db}{ds} + \frac{a}{\rho} + \frac{c}{\tau}, \quad \frac{\delta c}{ds} = \frac{dc}{ds} - \frac{b}{\tau}$$

If the point P remains fixed in space as M moves along the curve, the left-hand members of equations (82) are zero and the equations reduce to

$$(84) \quad \frac{d\xi}{ds} = \frac{\eta}{\rho} - 1, \quad \frac{d\eta}{ds} = -\left(\frac{\xi}{\rho} + \frac{\zeta}{\tau}\right), \quad \frac{d\zeta}{ds} = \frac{\eta}{\tau}$$

Moreover, the direction-cosines of a line fixed in space satisfy the equations

$$(85) \quad \frac{da}{ds} = \frac{b}{\rho}, \quad \frac{db}{ds} = -\left(\frac{a}{\rho} + \frac{c}{\tau}\right), \quad \frac{dc}{ds} = \frac{b}{\tau}$$

* Cf. Cesaro, *Lezioni di Geometria Intrinseca*, pp 122-128 Naples, 1896

are the Frenet-Serret formulas, as might have been

shall show that the solution of these equations carries with solution of (84) Suppose we have three sets of solutions $\alpha, l, \lambda, \beta, m, \mu, \gamma, n, \nu$, whose values for $s = 0$ are

$$1, 0, 0, 0, 1, 0, 0, 0, 1$$

are the direction-cosines, with respect to the moving trihedral vertex M , of three fixed directions in space mutually perpendicular to one another Let O be a fixed point, and through it three lines with the directions just found Take these lines as coordinate axes and let x, y, z denote the coordinates of M with respect to them If ξ, η, ζ denote the coordinates of O with respect to the moving trihedral, then $-\xi, -\eta, -\zeta$ are the direction-cosines of OM with respect to the trihedral with vertex at O and edges parallel to the corresponding edges of the trihedral at M Consequently we have

$$\xi = -(ax + \beta y + \gamma z),$$

$$\eta = -(lx + my + nz),$$

$$\zeta = -(\lambda x + \mu y + \nu z)$$

These values be substituted in (84) and we take account of (85), we find that the equations are identically satisfied If ξ_0, η_0, ζ_0 denote the values of ξ, η, ζ for $s = 0$, it follows from (87) that they differ only in sign from the initial values x_0, y_0, z_0 Hence if we write, in conformity with (21),

$$x = \int_0^s \alpha ds - \xi_0, \quad y = \int_0^s \beta ds - \eta_0, \quad z = \int_0^s \gamma ds - \zeta_0,$$

and substitute these values in (87), they become the general solution of the equations (84) We have seen that the solution of equations (84) reduces to the integration of the Riccati equation (65)

Illustrative examples As an example of the foregoing method we consider the locus of a point on the tangent to a twisted curve C at a distance a from the point of contact

Let M_1 be the point of the curve with reference to the axes at M

In this case equations (82) reduce to

$$\frac{\delta \xi}{\delta s} = 1, \quad \frac{\delta \eta}{\delta s} = \frac{a}{\rho}, \quad \frac{\delta \zeta}{\delta s} = 0$$

Hence if s_1 denotes the length of arc of C_1 from the point corresponding to $s =$ on C , we have

$$(u) \quad s_1 = \int_0^s \left(1 + \frac{a^2}{\rho^2}\right)^{\frac{1}{2}} ds,$$

and the direction-cosines of the tangent to C_1 with reference to the moving axes are given by

$$\alpha_1 = \frac{\rho}{\sqrt{a^2 + \rho^2}}, \quad \beta_1 = \frac{a}{\sqrt{a^2 + \rho^2}}, \quad \gamma_1 = 0$$

to C_1 is parallel to the osculating plane at the corresponding

(88) we find

$$\frac{d}{ds} \left(\frac{\rho}{\sqrt{a^2 + \rho^2}} \right) - \frac{a}{\rho \sqrt{a^2 + \rho^2}} = \frac{a^2 \rho'}{(a^2 + \rho^2)^{\frac{3}{2}}} - \frac{a}{\rho \sqrt{a^2 + \rho^2}}$$

in a manner with β_1 and γ_1 , and making use of (u), we have

$$\begin{aligned} \frac{\delta \alpha_1}{\delta s_1} &= \frac{a^2 \rho \rho'}{(a^2 + \rho^2)^2} - \frac{a}{a^2 + \rho^2}, & \frac{\delta \beta_1}{\delta s_1} &= \frac{-a \rho^2 \rho'}{(a^2 + \rho^2)^2} + \frac{\rho}{a^2 + \rho^2}, \\ \frac{\delta \gamma_1}{\delta s_1} &= -\frac{a \rho}{\tau (a^2 + \rho^2)} \end{aligned}$$

From these expressions and (27') we obtain the following expression for the square of the first curvature of C_1

$$(iii) \quad \frac{1}{\rho_1^2} = \frac{1}{a^2 + \rho^2} \left(\frac{a \rho \rho'}{a^2 + \rho^2} - 1 \right)^2 + \frac{a^2 \rho^2}{\tau^2 (a^2 + \rho^2)^2}$$

The direction cosines of the principal normal of C_1 are

$$l_1 = \rho_1 \frac{\delta \alpha_1}{\delta s_1}, \quad m_1 = \rho_1 \frac{\delta \beta_1}{\delta s_1}, \quad n_1 = \rho_1 \frac{\delta \gamma_1}{\delta s_1}$$

By means of (40) we derive the following expressions for the direction cosines of the binormal

$$\lambda_1 = -\frac{a^2 \rho \rho_1}{\tau (a^2 + \rho^2)^{\frac{3}{2}}}, \quad \mu_1 = \frac{a^2 \rho^2 \rho_1}{\tau (a^2 + \rho^2)^{\frac{3}{2}}}, \quad \nu_1 = \left(1 - \frac{a \rho \rho}{a^2 + \rho^2} \right) - \frac{\rho_1}{\sqrt{a^2 + \rho^2}}$$

In order to find the expression for τ_1 , the radius of torsion of C_1 we have only to substitute the above values in the equation

$$\frac{l_1}{\tau_1} = \frac{\delta \lambda_1}{\delta s_1} = \frac{\rho}{\sqrt{a^2 + \rho^2}} \left(\frac{d \lambda_1}{ds} - \frac{\mu_1}{\rho} \right)$$

We leave this calculation to the reader and proceed to an application of the preceding results

We inquire whether there is a curve C such that C_1 is a straight line. If necessary and sufficient condition is that $1/\rho_1$ be zero (Ex 3, p 15) From (iii) follows that we must have

$$\frac{a \rho \rho}{a^2 + \rho^2} - 1 = 0, \quad \frac{1}{\tau} = 0$$

second of these equations it follows that C must be plane, and from (1) we get, by integration,

$$\log(a^2 + \rho^2) = \frac{2s}{a} + c,$$

a constant of integration. If the point $s = 0$ be chosen so that we may have a^2 , this equation reduces to

$$\rho = a \sqrt{e^{\frac{2s}{a}} - 1}$$

where θ is the angle which the line C_1 makes with the ξ -axis, we have, from (1),

$$\tan \theta = \frac{\delta \eta}{\delta \xi} = \frac{a}{\rho} = \frac{1}{\sqrt{e^{\frac{2s}{a}} - 1}}$$

Integrating this equation with respect to s , we can put the result in the form

$$\frac{d\theta}{ds} = -\frac{1}{\rho},$$

hence (§ 6),

$$-\theta = \int \frac{ds}{\rho} = \sigma$$

When these values are substituted in equations (75), we obtain the coordinates of C in terms of θ ,

$$x = \int \sqrt{1 - e^{-\frac{2s}{a}}} ds, \quad y = ae^{-\frac{s}{a}},$$

hence of θ ,

$$x = -a \left[\log \tan \frac{\theta}{2} + \cos \theta \right], \quad y = a \sin \theta$$

The curve, with these equations, is called the *tractrix*. As just seen, it possesses the property that there is associated with it a straight line such that the segments between the points of tangency and points of intersection with the axis are of constant length.

11. *The orthogonal trajectories of the osculating planes of a twisted curve can be found by quadratures*

Reference to the moving axes the coordinates of a point in the osculating plane are $(\xi, \eta, 0)$. The necessary and sufficient condition that this point describe the orthogonal trajectory of the osculating plane is that it moves along the given curve and the $\frac{\delta \eta}{\delta s}$ in (82) be zero. Hence we have for the determination of ξ and η

$$\frac{d\xi}{d\sigma} - \eta + \rho = 0, \quad \frac{d\eta}{d\sigma} + \xi = 0,$$

given by (80). Eliminating ξ , we have

$$\frac{d^2 \eta}{d\sigma^2} + \eta = \rho$$

which can be found by quadratures as a function of σ , and consequently of s , and is given directly

Problem. Find a necessary and sufficient condition that a curve lie upon a sphere

If ξ, η, ζ denote the coordinates of the center, and R the radius of the sphere, we have $\xi^2 + \eta^2 + \zeta^2 = R^2$. Since the center is fixed, the derivatives of ξ, η, ζ are given by (84). Consequently, when we differentiate the above equation, the resulting equation reduces to $\xi = 0$, which shows that the normal plane to the curve at each point passes through the center of the sphere. If this equation be differentiated, we get $\eta = \rho$, hence the center of the sphere is on the polar line for each point. Another differentiation gives, together with the preceding, the following coordinates of the center of the sphere

$$(91) \quad \xi = 0, \quad \eta = \rho, \quad \zeta = -\tau\rho'$$

When the last of these equations is differentiated we obtain the desired condition

$$(92) \quad \frac{\rho}{\tau} + (\tau\rho')' = 0$$

Conversely, when this condition is satisfied, the point with the coordinates (91) is fixed in space and at constant distance from points of the curve. A curve which lies upon a sphere is called a *spherical curve*. Hence equation (92) is a necessary and sufficient condition that a curve be spherical.

EXAMPLES

1 Let C be a plane curve and C_1 an orthogonal trajectory of the normals to C . Show that the segments of these normals between C and C_1 are of the same length.

2 Let C and C_1 be two curves in the same plane, and say that the points correspond in which the curves are met by a line through a fixed point P . Show that if the tangents at corresponding points are parallel, the two curves are similar and P is the center of similitude.

3 The locus of the point of projection of a fixed point P upon the tangent to a curve C is called the *pedal curve* of C with respect to P . Show that if r is the distance from P to a point M on C , and θ the angle which the line PM makes with the tangent to C at M , the arc s_1 and radius of curvature ρ_1 of the pedal curve are given by

$$s_1 = \int \frac{r}{\rho} ds, \quad \rho_1 = \frac{r^2}{2r - \rho \sin \theta},$$

where s and ρ are the arc and the radius of curvature of C .

4 Find the intrinsic and parametric equations of a plane curve which is such that the segment on any tangent between the point of contact and the projection of a fixed point is of constant length.

5 Find the intrinsic equation of the plane curve which meets under constant angle all the lines passing through a fixed point.

6 The plane curve which is such that the locus of the mid-point of the segment of the normal between a point of the curve and the center of curvature is a straight line is the cycloid whose intrinsic equation is $\rho^2 + s^2 = a^2$.

7 Investigate the curve which is the locus of the point on the principal normal of a given curve and at constant distance from the latter.

ulating sphere Consider any curve whatever referred
 ving trihedral The point whose coordinates have the
 1) lies on the normal to the osculating plane at the
 curvature, that is, on the polar line Consequently
 ng sphere whose center is at this point, and whose
 $\sqrt{\rho^2 + \tau^2 \rho'^2}$, cuts the osculating plane in the osculating
 his sphere is called the *osculating sphere* to the curve at

We shall derive the property of this sphere which
 for its name

the tangent to a curve at a point M is tangent likewise
 e at this point, the center of the sphere lies in the normal
 he curve at M If R denotes its radius and the curve is
 o the trihedral at M , the coordinates of the center C of the
 e of the form $(0, y_1, z_1)$ and $y_1^2 + z_1^2 = R^2$ Let $P(x, y, z)$
 t of the curve near M , and Q the point in which the line
 he sphere If PQ be denoted by δ , we have, from (53),

$$\delta^2 = \left(s - \frac{1}{6\rho} s^3 + \right)^2 + \left(y_1 - \frac{1}{2\rho} s^2 + \frac{1}{6\rho^2} \rho' s^3 + \right)^2 \\
 + \left(z_1 + \frac{1}{6\tau\rho} s^3 + \right)^2,$$

lues to

$$2R\delta + \delta^2 = s^2 \left(1 - \frac{y_1}{\rho} \right) + \frac{s^3}{3\rho} \left(\frac{\rho'}{\rho} y_1 + \frac{z_1}{\tau} \right) +$$

is of the second order, in comparison with MP , unless
 at is, unless the center is on the polar line, then it is
 nd order unless $z_1 = -\rho'\tau$, in which case the sphere is
 ating sphere Hence we have the theorem

*osculating sphere to a curve at a point has contact with the
 the third order, other spheres with their centers on the
 , and tangent to the curve, have contact with the curve of
 l order, all other spheres tangent to the curve at a point
 act of the first order*

Thus of the osculating sphere is given by

$$R^2 = \rho^2 + \tau^2 \rho'^2,$$

oordinates of the center, referred to fixed axes in space, are

$$x = x + \rho l - \rho'\tau\lambda, \quad y_1 = y + \rho m - \rho'\tau\mu, \quad z_1 = z + \rho n - \rho'\tau\nu$$

Hence when ρ is constant the centers of the osculating sphere and the osculating circle coincide. Then the radius of the sphere is necessarily constant. Conversely, it follows from the equation (93)

that a necessary and sufficient condition that R be constant is

$$\rho'\tau\left[\frac{\rho}{\tau} + (\tau\rho')'\right] = 0,$$

that is, either the curvature is constant, or the curve is spherical.

If equations (94) be differentiated with respect to s , we get

$$(95) \quad \begin{cases} x'_1 = -\lambda \left[\frac{\rho}{\tau} + (\tau\rho')' \right], \\ y'_1 = -\mu \left[\frac{\rho}{\tau} + (\tau\rho')' \right], \\ z'_1 = -\nu \left[\frac{\rho}{\tau} + (\tau\rho')' \right] \end{cases}$$

From these expressions it is seen that the center of the osculating sphere is fixed only in case of spherical curves. Also, the tangent to the locus of the center is parallel to the binormal. Combining this result with a previous one, we have the theorem

The polar line for a point on a curve is tangent to the locus of the center of the osculating sphere to the curve at the corresponding point.

This result is represented in fig 5, in which the curve is the locus of the points M , the points C, C_1, C_2 are the corresponding centers of curvature, the planes $MCN, M_1C_1N_1$ are normal to the curve, the lines CP, C_1P_1 are the polar lines, and the points P, P_1, P_2 are the centers of the osculating spheres.

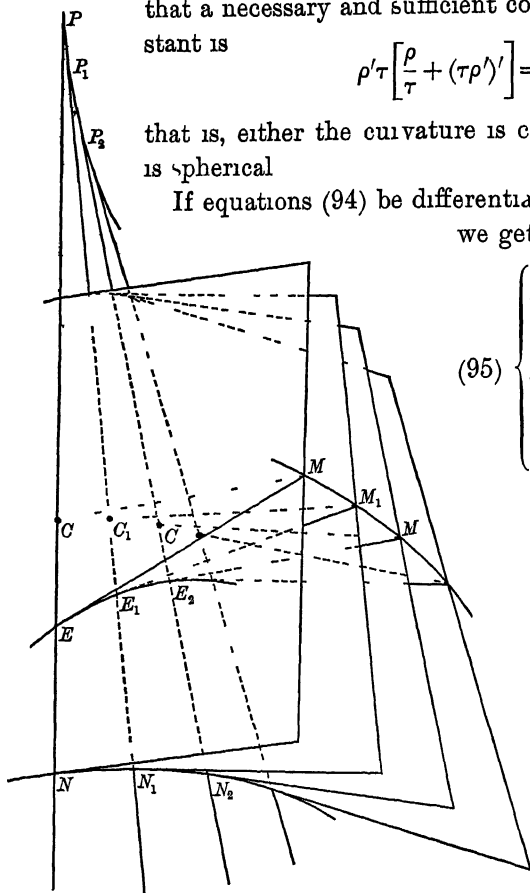


FIG 5

trand curves Bertrand proposed the following problem
*find the curves whose principal normals are the principal
 another curve* In solving this problem we make use of
 g trihedral We must find the necessary and sufficient
 that the point $M_1(\xi = 0, \eta = k, \zeta = 0)$ generate a curve C_1
 ncipal normal coincides with the η -axis of the moving
 Since the point M_1 remains on the moving η -axis, we
 $d\zeta = 0$ And since M_1 tends to move at right angles to
 $d\eta = 0$ Now equations (82) reduce to

$$\frac{\delta\xi}{ds} = 1 - \frac{k}{\rho}, \quad \frac{dk}{ds} = 0, \quad \frac{\delta\zeta}{ds} = -\frac{k}{\tau}$$

second we see that k is a constant Moreover, if ω denotes
 which the tangent at M_1 makes with the tangent at M ,
 from the first and third of these equations,

$$\tan \omega = \frac{\delta\zeta}{\delta\xi} = \frac{k\rho}{\tau(k - \rho)},$$

$$\frac{\sin \omega}{\rho} - \frac{\cos \omega}{\tau} = \frac{\sin \omega}{k}$$

e seen (§ 11) that according as τ is positive or negative,
 ating plane to a curve at a point M' near M cuts the
 for M below or above the osculating plane at M From
 siderations it follows that when $\tau > 0$, ω is in the third,
 first quadrants according as $k > \rho$, $0 < k < \rho$, or $k < 0$,
 $\tau < 0$, ω is in the second, first, or fourth quadrant,
 ly It is readily found that these results are consistent
 tion (97)

ns of (97) it is found from (96) that

$$\frac{\delta s_1}{ds} = \left[\left(\frac{\delta\xi}{ds} \right)^2 + \left(\frac{\delta\zeta}{ds} \right)^2 \right]^{\frac{1}{2}} = -\frac{k}{\tau \sin \omega},$$

ive sign being taken so that the left-hand member may
 e

ar we have expressed only the condition that the locus
 the moving η axis orthogonally, but not that this axis
 he principal normal to the curve C_1 also For this we
 the moving trihedral for C_1 and let a_1 , b_1 , c_1 denote the

direction-cosines with respect to it of a fixed direction in space, as M_1D in fig 6 They satisfy equations similar to (85), namely

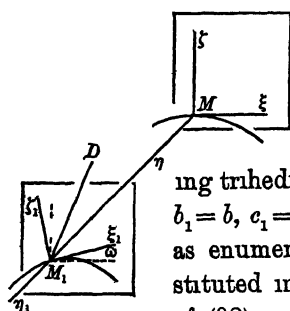


FIG 6

$$(99) \quad \frac{da_1}{ds_1} = \frac{b_1}{\rho_1}, \quad \frac{db_1}{ds_1} = -\left(\frac{a_1}{\rho_1} + \frac{c_1}{\tau_1}\right), \quad \frac{dc_1}{ds_1} = \frac{b_1}{\tau_1}$$

If a, b, c are the direction-cosines of the same direction, with respect to the moving trihedral at M , we must have $a_1 = a \cos \omega + c \sin \omega$, $b_1 = b$, $c_1 = -a \sin \omega + c \cos \omega$, for all possible cases, as enumerated above. When these values are substituted in the above equations, we get, by means of (98),

$$\begin{aligned} \left[\frac{\cos \omega}{\rho} + \frac{\sin \omega}{\tau} + \frac{k}{\rho_1 \tau \sin \omega} \right] b + (c \cos \omega - a \sin \omega) \frac{d\omega}{ds} &= 0, \\ \left[\frac{\tau \sin \omega}{k\rho} + \frac{\cos \omega}{\rho_1} - \frac{\sin \omega}{\tau_1} \right] a + \left[\frac{\sin \omega}{k} + \frac{\sin \omega}{\rho_1} + \frac{\cos \omega}{\tau_1} \right] c &= 0, \\ \left[\frac{\sin \omega}{\rho} - \frac{\cos \omega}{\tau} - \frac{k}{\tau_1 \tau \sin \omega} \right] b + (c \sin \omega + a \cos \omega) \frac{d\omega}{ds} &= 0 \end{aligned}$$

Since these equations must be true for every fixed line, the coefficients of a, b, c in each of these equations must be zero. The resulting equations of condition reduce to

$$(100) \quad \begin{cases} \omega = \text{const}, & \tau \tau_1 = \frac{k^2}{\sin^2 \omega}, \\ \frac{\sin \omega}{\rho_1} + \frac{\cos \omega}{\tau_1} + \frac{\sin \omega}{k} = 0 \end{cases}$$

Since ω is a constant, equation (97) is a linear relation between the first and second curvatures of the curve C . And the last of equations (100) shows that a similar relation holds for the curve C_1 .

Conversely, given a curve C whose first and second curvatures satisfy the relation

$$(101) \quad \frac{A}{\rho} + \frac{B}{\tau} = C,$$

where A, B, C are constants different from zero, if we take

$$k = \frac{A}{C}, \quad \cot \omega = -\frac{B}{A},$$

ρ_1 and τ_1 the values given by (100), equations (99) are satisfactorily, and the point $(0, k, 0)$ on the principal normal is the curve C_1 , conjugate to C . We gather these results on curves of Bertrand into the following theorem

Necessary and sufficient condition that the principal normals of two curves be the principal normals of a second is that a linear relation exist between the first and second curvatures, the distance between corresponding points of the two curves is constant, the osculating planes at these points cut under constant angle, and the torsions of the two curves have the same sign

Consider, finally, several particular cases, which we have in the consideration of equation (101)

$C = 0$ and $A \neq 0$, the ratio of ρ and τ is constant. Hence C is a helix and its conjugate is at infinity. When $A = 0$, when the curve has constant torsion, the conjugate curve coincides with the original. When $A = C = 0$, k is indeterminate, and curves admit of an infinity of conjugates, — they are all parallel to the given curve. The only other curve which has more than one conjugate is a circular helix, for since ρ and τ are constant, A/C can be given any value whatever, both a helix and the circular helices conjugate to it are traced on parallel cylinders with the same axis.

Tangent surface of a curve. For the further discussion of properties of curves it is necessary to introduce certain curves and surfaces which can be associated with them. However, in connection with these surfaces we limit our discussion to those properties which have to do with the associated curves, and leave other considerations to their proper places in later chapters.

Tangentiality of all the points on the tangents to a twisted curve C . Let S be the *tangent surface of the curve*. As thus defined, the surface consists of an infinity of straight lines, which are called the *generators* of the surface. Any point P on this surface lies on one of these lines, and is determined by this line and the distance t from the point M where the line touches the curve, as is shown in Fig. 1. If the coordinates x, y, z of M are expressed in terms of the arc length s , the coordinates of P are given by

$$\xi = x + x't, \quad \eta = y + y't, \quad \zeta = z + z't,$$

where the accents denote differentiation with respect to s . With the equations of the curve have the general form

$$x = f_1(u), \quad y = f_2(u), \quad z = f_3(u),$$

the coordinates of P can be expressed thus

$$(103) \quad \xi = f_1(u) + v f_1'(u), \quad \eta = f_2(u) + v f_2'(u), \quad \zeta = f_3(u) + v f_3'(u)$$

where

$$v = \frac{t}{[f_1'^2 + f_2'^2 + f_3'^2]^{\frac{1}{2}}}$$

From this it is seen that v is equal to the distance MP only when t is the parameter

As given by equations (102) or (103), the coordinates of a point on the tangent surface are functions of two parameters. A

relation between these parameters such as

$$(104) \quad f(s, t) = 0,$$

defines a curve which lies on the surface. For, when equation (104) is solved for t in terms of s and the resulting expression

is substituted in (102), the coordinates ξ, η, ζ are functions of a single parameter, and consequently

the locus of the point (ξ, η, ζ) is a curve (§ 1)

By definition, the element of arc of this curve is given by $d\sigma^2 = d\xi^2 + d\eta^2 + d\zeta^2$. This is expressible by means of (102) and (104) in the form

$$(105) \quad d\sigma^2 = \left(1 + \frac{t^2}{\rho^2}\right) ds^2 + 2 ds dt + dt^2,$$

where t is supposed to be the expression in s obtained from (104) and ρ is the radius of curvature of the curve C , of which the surface is the tangent surface. This result is true whatever be the relation (104). Hence equation (105) gives the element of arc of any curve on the surface, and $d\sigma$ is called the *linear element of the surface*.

According as t in equations (102) has a positive or negative value, the point lies on the portion of the tangent drawn in

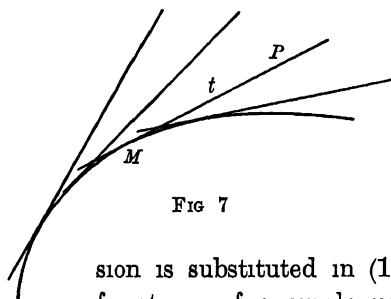


FIG 7

direction from the curve or in the opposite direction. It is our purpose to get an idea of the form of the surface in the neighborhood of the curve.

The sequence of (53) equations (102) can be written

$$\xi = \left(s - \frac{1}{6\rho^2}s^3 + \dots \right) + \left(1 - \frac{1}{2\rho^2}s^2 + \dots \right)t,$$

$$\eta = \left(\frac{s^2}{2\rho} - \frac{1}{6}\frac{\rho'}{\rho^2}s^3 + \dots \right) + \left(\frac{s}{\rho} - \frac{1}{2}\frac{\rho'}{\rho^2}s^2 + \dots \right)t,$$

$$\zeta = \left(-\frac{1}{6\rho\tau}s^3 + \dots \right) + \left(-\frac{1}{2\rho\tau}s^2 + \dots \right)t$$

The $\xi = 0$ cuts the surface in a curve Γ . The point M_0 of C , when $s = 0$, is also a point of Γ . From the above expression it is seen that for points of Γ near M_0 , the parameters s and t differ only in higher orders. Hence, neglecting powers of s and higher orders, the equations of Γ in the neighborhood of M_0 are

$$\xi = 0, \quad \eta = -\frac{t^2}{2\rho}, \quad \zeta = -\frac{1}{3\rho\tau}t^3$$

Eliminating t from the last two equations we find that in the neighborhood of M_0 the curve Γ has the form of a semi-parabola with the η axis, that is, the principal normal to C , for cuspidal point. Since any point of the curve C can be taken for M_0 , we have the theorem

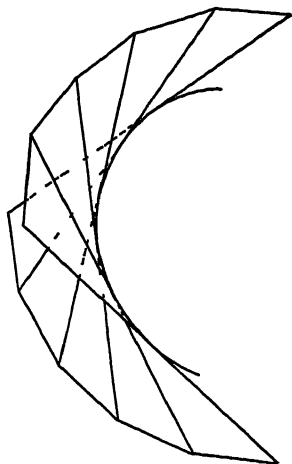


FIG. 8

tangent surface of a curve consists of two sheets, corresponding respectively to positive and negative values of t , which are tangent to each other along the curve, and thus form a sharp edge

For this account the curve is called the *edge of regression* of the surface.

An idea of the form of the surface may be had from fig. 8. **Involutes and evolutes of a curve.** When the tangents of a curve C are normal to a curve C_1 , the latter is called an *involute* of C .

C is called an *evolute* of C_1 . As thus defined, the involutes of a twisted curve lie upon its tangent surface, and those of a

plane curve in its plane. The latter is only a particular case of the former, so that the problem of finding the involutes of a curve is that of finding the curves upon the tangent surface which cut the generators orthogonally.

We write the equations of the tangent surface in the form

$$\xi = x + \alpha t, \quad \eta = y + \beta t, \quad \zeta = z + \gamma t$$

assuming that s is the parameter of the curve, the problem reduces to the determination of a relation between t and s such that

$$\alpha d\xi + \beta d\eta + \gamma d\zeta = 0$$

Use of (50) this reduces to $dt + ds = 0$, so that $t = c - s$, where c is an arbitrary constant. Hence the coordinates x_1, y_1, z_1 of the involute are expressible in the form

$$x_1 = x + \alpha(c - s), \quad y_1 = y + \beta(c - s), \quad z_1 = z + \gamma(c - s)$$

Corresponding to each value of c there is an involute, consequently a curve has an infinity of involutes. If two involutes correspond to values c_1 and c_2 of c , the segment of each tangent between the two curves is of length $c_1 - c_2$. Hence the involutes are said to form

a system of *parallel* curves on the tangent surface.

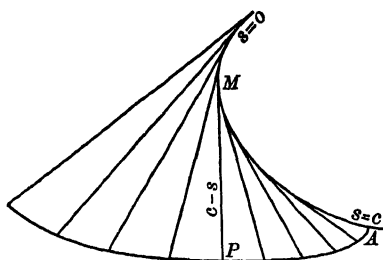


FIG. 9

When s is known the involutes are given directly by equation (106). Hence the complete determination of the involutes of a given curve requires one quadrature at most.

From the definition of t and its above value, an involute can be generated mechanically in the following manner, as represented in fig. 9. Take a string of length c and bring it into coincidence with the curve, with one end at the point $s = 0$, call the other end A . If the former point be fixed and the string be unwound gradually from the curve beginning at A , this point will trace out an involute on the tangent surface.

By differentiating equations (106), we get

$$dx_1 = \frac{l(c-s)}{\rho} ds, \quad dy_1 = \frac{m(c-s)}{\rho} ds, \quad dz_1 = \frac{n(c-s)}{\rho} ds$$

the tangent to an involute is parallel to the principal normal of the curve at the corresponding point, and consequently the normals at these points are perpendicular to one another

As an example of the foregoing theory, we determine the involutes of the cylinder, whose equations are

$$x = a \cos u, \quad y = a \sin u, \quad z = au \cot \theta,$$

where a is the radius of the cylinder and θ the constant angle which the tangent to the cylinder makes with the axis. Now

$$s = a \operatorname{cosec} \theta \quad u, \quad \alpha, \beta, \gamma = \frac{-\sin u, \cos u, \cot \theta}{\operatorname{cosec} \theta}$$

equations of the involutes are

$$x_1 = (au - c \sin \theta) \sin u, \quad y_1 = a \sin u - (au - c \sin \theta) \cos u, \quad z_1 = c \cos \theta$$

From the last of these equations it follows that the involutes are plane curves whose normals are normal to the axis of the cylinder, and from the expressions for α, β, γ it is seen that these curves are the involutes of the circular sections of the cylinder.

We now proceed to the inverse problem

to find the involutes of a curve C .

The problem reduces to the determination of a succession of normals to C which are tangent to a curve C_0 . If M_0 be the point on C_0 corresponding to M on C , it lies in the normal plane to C at M ; consequently its coordinates are of the form

$$x_0 = x + pl + q\lambda, \quad y_0 = y + pm + q\mu, \quad z_0 = z + pn + q\nu,$$

where p and q are the distances from M_0 to the binormal and principal normal respectively. These quantities p and q must be such that the line M_0M is tangent to the locus of M_0 at this point, that is, we must have

$$\frac{dx_0}{ds} = \kappa(x - x_0), \quad \frac{dy_0}{ds} = \kappa(y - y_0), \quad \frac{dz_0}{ds} = \kappa(z - z_0),$$

where κ denotes a factor of proportionality. When the above expressions are substituted in these equations, we get

$$\alpha \left(1 - \frac{p}{\rho} \right) + l \left(\frac{dp}{ds} + \frac{q}{\tau} + p\kappa \right) + \lambda \left(\frac{dq}{ds} - \frac{p}{\tau} + q\kappa \right) = 0,$$

and other equations obtained by replacing α, l, λ by β, m, μ and γ, n, ν . Hence the expressions in parentheses vanish. From

the first it follows that p is equal to ρ , consequently M_0 lies on the polar line of C at M . The other equations of condition can be written

$$\frac{d\rho}{ds} + \frac{q}{\tau} + \rho\kappa = 0, \quad \frac{dq}{ds} - \frac{\rho}{\tau} + q\kappa = 0$$

Eliminating κ , we get

$$\frac{ds}{\tau} = \frac{\rho dq - q d\rho}{\rho^2 + q^2} = d \tan^{-1} \frac{q}{\rho}$$

For the sake of convenience we put $\omega = \int \frac{ds}{\tau}$, and obtain by integration

$$\frac{q}{\rho} = \tan(\omega + c),$$

where c is the constant of integration. As c is arbitrary, there is an infinity of evolutes of the curve C , they are defined by the following equations, in which c is constant for an evolute but changes with it

$$\begin{aligned} x_0 &= x + l\rho + \lambda\rho \tan(\omega + c), & y_0 &= y + m\rho + \mu\rho \tan(\omega + c), \\ z_0 &= z + n\rho + \nu\rho \tan(\omega + c) \end{aligned}$$

From the definition of q it follows that q/ρ is equal to the tangent of the angle which MM_0 makes with the principal normal to C at M . Calling this angle θ , we have $\theta = \omega + c$. The foregoing results give the following theorem

A curve C admits of an infinity of evolutes, when each of the normals to C , which are tangent to one of its evolutes, is turned through the same angle in the corresponding normal plane to C , these new normals are tangent to another evolute of C .

In fig 5 the locus of the points E is an evolute of the given curve

Each system of normals to C which are tangent to an evolute C_0 constitute a tangent surface of which C_0 is the edge of regression. Hence the evolutes of C are the edges of regression of an infinity of tangent surfaces, all of which pass through C .

From the definition of ω it follows that ω is constant only when the curve C is plane. In this case we may take ω equal to zero. Then when $c = 0$ we have the evolute C_0 in the plane of the curve. The other evolutes lie upon the right

rmed of the normals to the plane at points of C_0 , and cut the elements under under the constant angle $90^\circ - c$, and consequently are helices have the theorem

lutes of a plane curve are the helices traced on the right cylinder whose plane evolute Conversely, *every cylindrical helix is the evolute of an plane curves*

EXAMPLES

the coordinates of the center of the osculating sphere of the twisted

angle between the radius of the osculating sphere for any curve and the center of the sphere is equal to the angle between the radius of the circle and the locus of the center of curvature

locus of the center of curvature of a curve is an evolute only when the and

the radii of first and second curvature of the curve $x = a \sin u \cos u$, u , $z = a \sin u$ Show that the curve is spherical, and give a geometrical on Find its evolutes.

ive the properties of Bertrand curves (§ 19) without the use of the moving

l the involutes and evolutes of the twisted cubic

ermine whether there is a curve whose binormals are the binormals of a rve

ive the results of § 21 by means of the moving trihedral

Minimal curves In the preceding discussion we have made on of the curves, defined by

$$x = f_1(u), \quad y = f(u), \quad z = f_3(u),$$

these functions satisfy the condition

$$f_1'^2 + f_2' + f_3' = 0$$

e imaginary curves are of interest in certain parts of the of surfaces, we devote this closing section to their discussion equation of condition may be written in the form

$$\frac{f_1' + if_2'}{-f_3'} = \frac{f_3'}{f_1' - if_2'} = v,$$

v is a constant or a function of u These equations are ent to the following

$$f_1' f_2' f_3' = \frac{1-v^2}{2} \cdot \frac{(1+v^2)}{2} v$$

At most, the common ratio is a function of u , say $f(u)$. At if we disregard additive constants of integration, as they can be removed by a translation of the curve in space, we can re-write the above equations by

$$(109) \quad x = \int \frac{1-v^2}{2} f(u) du, \quad y = \int \frac{1+v^2}{2} f(u) du, \quad z = \int v f(u) du$$

We consider first the case when v is constant and call it a . If we change the parameter of the curve by replacing $\int f(u) du$ by a new parameter which we call u , we have, without loss of generality,

$$(110) \quad x = \frac{1-a^2}{2} u, \quad y = \frac{1+a^2}{2} u, \quad z = au$$

For each value of a these are the equations of an imaginary straight line through the origin. Eliminating a , we find that the envelope of these lines is the imaginary cone, with vertex at the origin, whose equation is

$$(111) \quad x^2 + y^2 + z^2 = 0$$

Every point on the cone is at zero distance from the vertex, for from the equations of the lines it is seen that the distance between any two points on a line is zero. We call these generators of the cone *minimal straight lines*. Through any point in space there are an infinity of them, their direction-cosines are proportional to

$$\frac{1-a^2}{2}, \quad \frac{1+a^2}{2}, \quad a,$$

where a is arbitrary. The locus of these lines is the cone whose vertex is at the point and whose generators pass through the point at infinity. For, the equation in homogeneous coordinates of the sphere of unit radius and center at the origin is $x^2 + y^2 + z^2 = 0$ so that the equations of the circle at infinity are

$$x^2 + y^2 + z^2 = 0, \quad w = 0$$

Hence the cone (111) passes through the circle at infinity.

We consider now the case where v in equations (109) is a function of u . If we take this function of u for a new parameter, and for convenience call it u , equations (109) may be written in the form

$$(112) \quad x = \int \frac{1-u^2}{2} F(u) du, \quad y = \int \frac{1+u^2}{2} F(u) du, \quad z = \int u F(u) du$$

is seen from (108), $F(u)$ can be any function of u different from zero

replace $F(u)$ by the third derivative of a function $f(u)$, $F(u) = f'''(u)$, equations (112) can be integrated by parts and put in the form

$$\begin{cases} x = \frac{1}{2}(1-u^2)f''(u) + uf'(u) - f(u), \\ y = \frac{i}{2}(1+u^2)f''(u) - uf'(u) + if(u), \\ z = uf''(u) - f'(u) \end{cases}$$

must be different from zero, $f(u)$ can have any form other than $a^2 + c_1u + c_2$, where c_1, c_2, c_3 are arbitrary constants

EXAMPLES

show that the tangents to a minimal curve are minimal lines, and that a surface whose tangents are minimal lines is minimal

show that the osculating plane of a minimal curve can be written $(X-x)A + (Y-y)B + (Z-z)C = 0$, where $A^2 + B^2 + C^2 = 0$. A plane whose equation is thus called an *isotropic plane*

show that through each point of a plane two minimal straight lines pass, and that in the latter case the plane is isotropic

determine the order of the minimal curves for which the function f in (113) satisfies the condition $4f''f''' - 5f''^2 = 0$

show that the equations of a minimal curve, for which f in (113) satisfies the condition $4f''f''' - 5f''^2 = af'^3$, where a is a constant, can be put in the form

$$x = \frac{8}{a} \cos t, \quad y = \frac{8}{a} \sin t, \quad z = \frac{8i}{a} t$$

GENERAL EXAMPLES

show that the equations of any plane curve can be put in the form

$$x = \int \cos \phi f(\phi) d\phi, \quad y = \int \sin \phi f(\phi) d\phi,$$

determine the geometrical significance of ϕ

show that the necessary and sufficient condition that the parameter u in the equations $x = f_1(u)$, $y = f_2(u)$ have the significance of ϕ in the preceding example is

$$f_1 f_2' - f_1' f_2 = f_1^2 + f_2^2$$

show that the general projective transformation transforms an osculating plane of a curve into an osculating plane of the transform

show that the principal normal to a curve is normal to the locus of the centers of curvature at the points where ρ is a maximum or minimum

5 A certain plane curve possesses the property that if C be its center of curvature for a point P , Q the projection of P on the x -axis, and T the point where the tangent at P meets this axis, the area of the triangle CQT is constant. Find the equations of the curve in terms of the angle which the tangent forms with the x -axis.

6 The binormal at a point M of a curve is the limiting position of the common perpendicular to the tangents at M and M' , as M' approaches M .

7 The tangents to the spherical indicatrices of the tangent and binormal of a twisted curve at corresponding points are parallel.

8 Any curve upon the unit sphere serves for the spherical indicatrix of the binormal of a curve of constant torsion. Find the coordinates of the curve.

9 The equations

$$x = a \int \frac{l dk - k dl}{h^2 + k^2 + l^2}, \quad y = a \int \frac{h dl - l dh}{h^2 + k^2 + l^2}, \quad z = a \int \frac{k dh - h dk}{h^2 + k^2 + l^2},$$

where a is constant and h, k, l are functions of a single parameter, define a curve whose radius of torsion is a .

10 If, in Ex. 9, we have

$$h = \cos \mu \theta - \sqrt{\frac{\mu}{\lambda}} \cos \lambda \theta, \quad k = \sin \mu \theta + \sqrt{\frac{\mu}{\lambda}} \sin \lambda \theta, \quad l = 2 \left(\frac{\mu}{\lambda} \right)^{\frac{1}{2}} \cos \frac{\lambda + \mu}{2} \theta,$$

where λ and μ are constants whose ratio is commensurable, the integrands are expressible as linear homogeneous functions of sines and cosines of multiples of θ , and consequently the curve is algebraic.

11 Equations (1) define a family of circles, if a, b, r are functions of a parameter t . Show that the determination of their orthogonal trajectories requires the solution of the Riccati equation,

$$\frac{d\theta}{dt} = \frac{1}{r} \frac{da}{dt} \theta - \frac{1}{2r} \frac{db}{dt} (1 - \theta^2),$$

where $\theta = \tan u/2$.

12 Find the vector representing the rate of change of the acceleration of a moving point.

13 When a curve is spherical, the center of curvature for the point is the foot of the perpendicular upon the osculating plane at the point from the center of the sphere.

14 The radii of first and second curvature of a curve which lies upon a sphere and cuts the meridians under constant angle are in the relation $1 + a\tau + b\rho^2\tau = 0$, where a and b are constants.

15 An *epitrochoidal curve* is generated by a point in the plane of a circle which rolls, without slipping, on another circle, whose plane meets the plane of the first circle under constant angle. Find its equations and show that it is a spherical curve.

16 If two curves are in a one to one correspondence with the tangents at corresponding points parallel, the principal normals at these points are parallel and likewise the binormals, two curves so related are said to be deducible from one another by a *transformation of Combescure*.

17 If two curves are in a one to one correspondence and the osculating planes at corresponding points are parallel, either curve can be obtained from the other by a transformation of Combescure.

show that the radius of the osculating sphere of a curve is given by $\frac{(\rho''^2 + y'''^2 + z''^2) - \tau^2}{\rho^2}$, where the prime denotes differentiation with respect to the arc

corresponding points of a twisted curve and the locus of the center of the osculating sphere the principal normals are parallel, and the tangent to one curve is parallel to the binormal to the other also the product of the radii of the two curves is equal to the product of the radii of first curvature, with the sign, according as the positive directions of the principal normals are the same or different

determine the twisted curves which are such that the centers of the spheres of osculation of the curve of centers of the osculating spheres of the given curve are points on the given curve

show that the binormals to a curve do not constitute the tangent surface of the curve

determine the directions of the principal normal and binormal to an involute curve

show that the equations

$$a \int \phi(u) \sin u \, du, \quad y = a \int \phi(u) \cos u \, du, \quad z = a \int \phi(u) \psi(u) \, du,$$

where $\phi(u) = (1 + \psi'^2 + \psi'^2)^{\frac{1}{2}}(1 + \psi^2)^{-\frac{1}{2}}$ and $\psi(u)$ is any function whatever, define a curve of constant curvature

show that when $\psi(u) = \tan u$, in example 23, the curve is algebraic

show that in order that the principal normals of a curve be the binormals to another, the relation $a\left(\frac{1}{\rho^2} + \frac{1}{\tau^2}\right) = \frac{b}{\rho}$ must hold, where a and b are constants

show that such curves are defined by equations of example 23 when

$$\phi = \frac{(1 + \psi' + \psi)^3 + (1 + \psi)^3(\psi' + \psi)^3}{(1 + \psi^2)^{\frac{1}{2}}(1 + \psi' + \psi)^{\frac{1}{2}}}$$

let λ_1, μ_1, ν_1 be the coordinates of a point on the unit sphere expressed in terms of the arc σ_1 of the curve. Show that the equations

$$x = ek \int \lambda_1 d\sigma_1 - k \cot \omega \int (\mu_1 \nu_1 - \mu_1 \nu_1) d\sigma_1,$$

$$y = ek \int \mu_1 d\sigma_1 - k \cot \omega \int (\nu_1 \lambda_1 - \nu_1 \lambda_1) d\sigma_1,$$

$$z = ek \int \nu_1 d\sigma_1 - k \cot \omega \int (\lambda_1 \mu_1 - \lambda_1 \mu_1) d\sigma_1,$$

and ω are constant, $e = \pm 1$, and the primes indicate differentiation with respect to σ_1 , define a Bertrand curve for which ρ and τ satisfy the relation (97) and that λ_1, μ_1, ν_1 are the direction cosines of the binormal to the conjugate

CHAPTER II

CURVILINEAR COORDINATES ON A SURFACE ENVELOPES

23 Parametric equations of a surface In the preceding chapter we have seen that the coordinates of a point on the tangent surface of a curve are expressible in the form

$$(1) \quad x = f_1(u) + v f_1'(u), \quad y = f_2(u) + v f_2'(u), \quad z = f_3(u) + v f_3'(u),$$

$$\text{where} \quad \xi = f_1(u), \quad \eta = f_2(u), \quad \zeta = f_3(u),$$

are the equations of the curve, and v is proportional to the distance between the points (ξ, η, ζ) , (x, y, z) on the same generator. Since the coordinates of the surface are expressed by (1) as functions of

two independent *parameters* u, v , the *equations of the surface* may be written

$$(2) \quad \begin{cases} x = f_1(u, v), \\ y = f_2(u, v), \\ z = f_3(u, v) \end{cases}$$

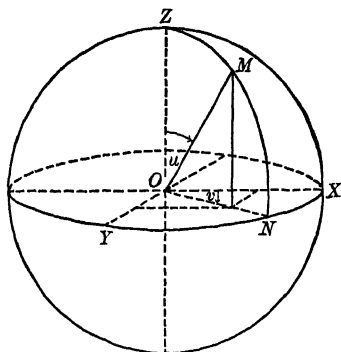


FIG 10

Consider also a sphere of radius a whose center is at the origin O (fig 10). If v denotes the angle, measured in the positive sense, which the plane through the z axis and a point M of the sphere makes

with the xz plane, and u denotes the angle between the radius OM and the positive z -axis, the coordinates of M may be written

$$(3) \quad x = a \sin u \cos v, \quad y = a \sin u \sin v, \quad z = a \cos u$$

Here, again, the coordinates of any point on the sphere are expressible as functions of two parameters, and the *equations of the sphere* are of the form (2)*

* Notice that in this case f_3 is a function of u alone

the two preceding cases the functions f_1, f_2, f_3 have particular forms. We consider the general case where f_1, f_2, f_3 are functions of two independent parameters u, v , analytic for all values of u and v , or at least for values within a certain domain around the point whose coordinates are given by (2) for all values of u and v in the domain is called a *surface*. And equations (2) are called *parametric equations of the surface*.

It is to be understood that one or more of the functions f may be functions of a single parameter. For instance, any cylinder may be represented by equations of the form

$$x = f_1(u), \quad y = f_2(u), \quad z = f_3(u, v)$$

where we replace u and v in (2) by independent functions of two parameters u_1, v_1 , thus

$$u = F_1(u_1, v_1), \quad v = F_2(u_1, v_1),$$

the resulting equations may be written

$$x = \phi_1(u_1, v_1), \quad y = \phi_2(u_1, v_1), \quad z = \phi_3(u_1, v_1)$$

When particular values of u_1 and v_1 be substituted in (4) and the resulting values of u and v be substituted in (2), we obtain the values of x, y, z given by (5), when u_1 and v_1 have been given the particular values. Hence equations (2) and (5) define the same surface, provided that F_1 and F_2 are of such a form that ϕ_1, ϕ_2, ϕ_3 satisfy the general conditions imposed upon the F 's. Hence the existence of a surface may be expressed in parametric form in a number of ways of the generality of two arbitrary functions. Suppose the first two of equations (2) solved for u and v in terms of x and y , and let $u = F_1(x, y), v = F_2(x, y)$ be a set of solutions. When these equations are taken as equations (4), equation (5) becomes

$$x = x, \quad y = y, \quad z = f(x, y),$$

which may be replaced by the single relation,

$$z = f(x, y)$$

which is only one set of solutions of the first two of equations (2). Equation (6) defines the surface as completely as (2). If, however, we have n sets of solutions, the surface would be defined by n equations, $z = f_1(x, y)$

where a and b denote constants. Unless u or v is absent from either of these equations the curves are necessarily distinct from the parametric curves $u = \text{const}$ and $v = \text{const}$. Suppose, now, that v does not appear in Φ_1 , then u_1 is constant when u is constant, and vice versa. Consequently a curve $u_1 = \text{const}$ is a member of the family of curves $u = \text{const}$. Hence, when a transformation of parameters is made by means of equations of the form

$$u_1 = \Phi_1(u), \quad v_1 = \Phi_2(v),$$

or

$$u_1 = \Phi_1(v), \quad v_1 = \Phi_2(u),$$

the two systems of parametric curves are the same, the difference being in the value of the parameter which is constant along a curve

right conoid

EXAMPLES

✕ 1 A surface which is the locus of a family of straight lines, which meet another straight line orthogonally and are arranged according to a given law, is called a *right conoid*, its equations are of the form $x = u \cos v$, $y = u \sin v$, $z = \phi(v)$. Show that when $\phi(v) = a \cot v + b$ the conoid is a hyperbolic paraboloid

✕ 2 Find the equations of the right conoid whose axis is the axis of z , and which passes through the ellipse $x = a$, $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

3 When a sphere of radius a is defined by (3), find the relation between u and v along the curve of intersection of the sphere and the surface $x^4 + y^4 + z^4 = \frac{1}{2}a^4$. Show that the curves of intersection are four great circles

✕ 4 Upon the surface $x = \sqrt{u^2 + \frac{1}{2}} \cos v$, $y = \sqrt{u^2 + \frac{1}{2}} \sin v$, $z = u$, determine the curves whose tangents make with the z -axis the angle $\tan^{-1} \sqrt{2}$. Show that two of these curves pass through every point, and find their radii of first and second curvature

25 Tangent plane A tangent line to a curve upon a surface is called a *tangent line to the surface* at the point of contact. It is evident that there are an infinity of tangent lines to a surface at a point. We shall show that all of these lines lie in a plane, which is called the *tangent plane* to the surface at the point.

To this end we consider a curve C upon a surface and let $M(x, y, z)$ be the point at which the tangent is drawn. The equations of the tangent are (§ 4)

$$\frac{\xi - x}{\frac{dx}{ds}} = \frac{\eta - y}{\frac{dy}{ds}} = \frac{\zeta - z}{\frac{dz}{ds}} = \lambda,$$

η, ζ are the coordinates of a point on the line, depending values upon the parameter λ . If the equation in curvilinear coordinates of the curve C is $v = \phi(u)$, the above equations written

$$\xi - x = \lambda \left(\frac{\partial x}{\partial u} + \phi' \frac{\partial x}{\partial v} \right) \frac{du}{ds},$$

$$\eta - y = \lambda \left(\frac{\partial y}{\partial u} + \phi' \frac{\partial y}{\partial v} \right) \frac{du}{ds},$$

$$\zeta - z = \lambda \left(\frac{\partial z}{\partial u} + \phi' \frac{\partial z}{\partial v} \right) \frac{du}{ds},$$

the prime indicates differentiation. In order to obtain these tangent lines, we eliminate ϕ' and λ from these. This gives

$$\begin{vmatrix} \xi - x & \eta - y & \zeta - z \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = 0,$$

which is evidently the equation of a plane through the point M . The line normal to this plane at the point of contact is called the normal to the surface at the point.

For example, we find the equation of the tangent plane to the tangent surface at any point. If the values from (1) be substituted in equation (11), the resulting equation is reducible to

$$\begin{vmatrix} \xi - f_1 & \eta - f_2 & \zeta - f_3 \\ f_1 & f_2 & f_3 \\ f_1 & f_2 & f_3 \end{vmatrix} = 0$$

the equation of the tangent plane is independent of v , and depends only on the consequence of (I, 36)* we have the theorem

The tangent plane to the tangent surface of a curve is the same at all points of it as it is the osculating plane of the curve at the point where the generator curve

the surface is defined by an equation of the form $F(x, y, z) = 0$, we treat x, y, z as functions of u and v , and differentiate with respect to

This gives

$$\frac{\partial F}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial u} = 0, \quad \frac{\partial F}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial v} = 0$$

* In references of this sort the Roman numerals refer to the chapter

58 CURVILINEAR COORDINATES ON A SURFACE

By means of these equations the equation (11) of the tangent plane can be given the form

$$(\xi - x) \frac{\partial F}{\partial x} + (\eta - y) \frac{\partial F}{\partial y} + (\zeta - z) \frac{\partial F}{\partial z} = 0$$

When the Monge form of the equation of a surface, namely $z = f(x, y)$, is used, it is customary to put

$$(14) \quad \frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q$$

Consequently the equation of the tangent plane is

$$(15) \quad (\xi - x)p + (\eta - y)q - (\zeta - z) = 0$$

In the first chapter we found that a curve is defined by two equations of the form

$$(16) \quad F_1(x, y, z) = 0, \quad F_2(x, y, z) = 0$$

Hence a curve is the locus of the points common to two surfaces. The equations of the tangent to the curve are

$$\frac{\xi - x}{dx} = \frac{\eta - y}{dy} = \frac{\zeta - z}{dz},$$

where dx, dy, dz satisfy the relations

$$\frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy + \frac{\partial F_1}{\partial z} dz = 0, \quad \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz = 0$$

Consequently the equations of the tangent can be put in the form

$$(17) \quad \frac{\xi - x}{\frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial y}} = \frac{\eta - y}{\frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial z}} = \frac{\zeta - z}{\frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial x}}$$

Comparing this result with (13), we see that the tangent line to a curve at a point M is the intersection of the tangent planes at M to two surfaces which intersect along the curve.

EXAMPLES

✓ 1 Show that the volume of the tetrahedron formed by the coordinate planes and the tangent plane at any point of the surface $x = u, y = v, z = a^3/uv$ is constant.

2 Show that the sum of the squares of the intercepts of the axes by the tangent plane to the surface

$$x = u^3 \sin^3 v, \quad y = u^3 \cos^3 v, \quad z = (a^2 - u^2)^{\frac{3}{2}},$$

at any point is constant.

3 Given the right conoid for which $\phi(v) = a \sin 2v$. Show that any tangent plane to the surface cuts it in an ellipse, and that if perpendiculars be drawn to the generators from any point the feet of the perpendiculars lie in a plane ellipse.

that the tangent planes, at points of a generator, to the right conoid for $a \sqrt{\tan v}$, meet the plane $z = 0$ in parallel lines.

The equations of the tangent to the curve whose equations are

$$ax^2 + by^2 + cz^2 = 1, \quad bx^2 + cy^2 + az^2 = 1$$

The equations of the tangent to the curve whose equations are

$$z(x+z)(x-a) = a^3, \quad z(y+z)(y-a) = a^3,$$

at the curve is plane

distance from a point M' of a surface to the tangent plane at a near-by point M of the second order when MM' is of the first order, and for other planes the distance from M' is ordinarily of the first order

-parameter families of surfaces Envelopes An equation in

$$F(x, y, z, a) = 0$$

infinity of surfaces, each surface being determined by a value of the parameter a . Such a system is called a *one parameter family of surfaces*. For example, the tangent planes to the tangent planes of a twisted curve form such a family.

Two surfaces corresponding to values a and a' of the parameter a in a curve whose equations may be written

$$F(x, y, z, a) = 0, \quad \frac{F(x, y, z, a') - F(x, y, z, a)}{a' - a} = 0$$

As a approaches a' , this curve approaches a limiting form whose equation is

$$F(x, y, z, a) = 0, \quad \frac{\partial F(x, y, z, a)}{\partial a} = 0$$

The curve thus defined is called the *characteristic* of the surface of the family. As a varies we have a family of these characteristics, the locus, called the *envelope* of the family of surfaces, whose equation is obtained by eliminating a from the two equations (19). This elimination may be accomplished by solving (19) for a , thus

$$a = \phi(x, y, z),$$

Substituting in the first with the result

$$F(x, y, z, \phi) = 0$$

The equation of the tangent plane to this surface is

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial \phi} \frac{\partial \phi}{\partial x}\right)(\xi - x) + \left(\frac{\partial F}{\partial y} + \frac{\partial F}{\partial \phi} \frac{\partial \phi}{\partial y}\right)(\eta - y) + \left(\frac{\partial F}{\partial z} + \frac{\partial F}{\partial \phi} \frac{\partial \phi}{\partial z}\right)(\zeta - z) = 0$$

icular value of a , say a_0 , equations (19) define the curve where the surface $F(x, y, z, a_0) = 0$ meets the envelope, and from (19) it follows that at all points of this curve equation of the tangent plane to the envelope reduces to

$$\frac{\partial F}{\partial x}(\xi - x) + \frac{\partial F}{\partial y}(\eta - y) + \frac{\partial F}{\partial z}(\zeta - z) = 0$$

is, however, is the equation of the tangent plane to the surface $F(x, y, z, a_0) = 0$. If we say that two surfaces with the same tangent plane at a common point are tangent to one another, we have

The envelope of a family of surfaces of one parameter is tangent to each surface along the characteristic of the latter

The equations of the characteristic of the surface of parameter a_1 are

$$(21) \quad F(x, y, z, a_1) = 0, \quad \left(\frac{\partial F}{\partial a}\right)_{a=a_1} = 0$$

This characteristic meets the characteristic (19) in the point whose coordinates satisfy (19) and (21), or, what is the same thing, equations (19) and

$$\frac{F(x, y, z, a_1) - F(x, y, z, a)}{a_1 - a} = 0, \quad \frac{\left(\frac{\partial F}{\partial a}\right)_{a=a_1} - \left(\frac{\partial F}{\partial a}\right)_{a=a}}{a_1 - a} = 0$$

As a_1 approaches a , this point of intersection approaches a limiting position whose coordinates satisfy the three equations

$$(22) \quad F = 0, \quad \frac{\partial F}{\partial a} = 0, \quad \frac{\partial^2 F}{\partial a^2} = 0$$

If these equations be solved for x, y, z , we have

$$(23) \quad x = f_1(a), \quad y = f_2(a), \quad z = f_3(a)$$

These are parametric equations of a curve, which is called the *edge of regression* of the envelope

direction-cosines of the tangent to the edge of regression proportional to $\frac{dx}{da}, \frac{dy}{da}, \frac{dz}{da}$. If we imagine that x, y, z in (19) are replaced by the values (23), and we differentiate these equations with respect to a , we get, in consequence of (22),

$$\begin{aligned} \frac{\partial F}{\partial x} \frac{dx}{da} + \frac{\partial F}{\partial y} \frac{dy}{da} + \frac{\partial F}{\partial z} \frac{dz}{da} &= 0, \\ \frac{\partial^2 F}{\partial a \partial x} \frac{dx}{da} + \frac{\partial^2 F}{\partial a \partial y} \frac{dy}{da} + \frac{\partial^2 F}{\partial a \partial z} \frac{dz}{da} &= 0 \end{aligned}$$

hence we obtain

$$\frac{dx}{da} \frac{dy}{da} \frac{dz}{da} = \left\| \begin{array}{ccc} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial^2 F}{\partial a \partial x} & \frac{\partial^2 F}{\partial a \partial y} & \frac{\partial^2 F}{\partial a \partial z} \end{array} \right\|$$

in (17) it follows that the minors of the right-hand member are proportional to the direction-cosines of the tangent to the edge of regression. Hence we have the theorem

Characteristics of a family of surfaces of one parameter are tangent to the edge of regression

Developable surfaces Rectifying developable A simple example of a family of surfaces of one parameter is afforded by a family of planes of one parameter. Their envelope is called a *developable surface*, the full significance of this term will be explained later (§ 43). The characteristics are straight lines which are tangent to a curve, the edge of regression. When the edge of regression is a point, the surface is a cone or cylinder, according as the point is at a finite or infinite distance. We exclude this case for the present and assume that the coordinates x, y, z of a point on the edge of regression are expressed in terms of the arc s . We may write the equation of the plane

$$(X-x)a + (Y-y)b + (Z-z)c = 0,$$

where x, y, z are functions of s . The characteristics are defined by the equation and its derivative with respect to s , namely

$$(X-x)a' + (Y-y)b' + (Z-z)c' - ax' - by' - cz' = 0$$

Since these equations define the tangent to the curve, they may be equivalent to the equations

$$\frac{X-x}{x'} = \frac{Y-y}{y'} = \frac{Z-z}{z'}$$

Hence we must have

$$(26) \quad ax' + by' + cz' = 0, \quad a'x' + b'y' + c'z' = 0$$

If the first of these equations be differentiated with respect to the resulting equation is reducible, in consequence of the second of (26), to

$$ax'' + by'' + cz'' = 0$$

From this equation and (26) we find

$$a \ b \ c = (y'z'' - z'y'') \ (z'x'' - x'z'') \ (x'y'' - y'x'')$$

Hence by (§ 7) we have the theorem

On the envelope of a one-parameter family of planes the planes osculate the edge of regression

We leave it to the reader to prove that the edge of regression of the osculating planes of a twisted curve is the curve itself

The envelope of the plane normal to the principal normal of a curve at a point of the curve is called the *rectifying developable* of the latter. We shall find the equations of its edge of regression

The equation of this plane is

$$(27) \quad (X-x)l + (Y-y)m + (Z-z)n = 0$$

If we differentiate this equation with respect to the arc of the curve and make use of the Frenet formulas (I, 50), we obtain

$$(28) \quad (X-x)\left(\frac{\alpha}{\tau} + \frac{\lambda}{\tau}\right) + (Y-y)\left(\frac{\beta}{\rho} + \frac{\mu}{\tau}\right) + (Z-z)\left(\frac{\gamma}{\rho} + \frac{\nu}{\tau}\right) = 0$$

From these equations we derive the equations of the characteristic in the form

$$X = x + \left(\frac{\alpha}{\tau} - \frac{\lambda}{\rho}\right)t, \quad Y = y + \left(\frac{\beta}{\tau} - \frac{\mu}{\rho}\right)t, \quad Z = z + \left(\frac{\gamma}{\tau} - \frac{\nu}{\rho}\right)t,$$

the parameter of points on the characteristic. In order to value of t corresponding to the point where the characteristic touches the edge of regression, we combine these equations and derivative of (28) with respect to s , namely

$$\frac{\alpha\rho'}{\rho^2} + \frac{\lambda\tau'}{\tau^2} + (Y-y)\left[\frac{\beta\rho'}{\rho^2} + \frac{\mu\tau'}{\tau^2}\right] + (Z-z)\left[\frac{\gamma\rho'}{\rho^2} + \frac{\nu\tau'}{\tau^2}\right] + \frac{1}{\rho} = 0,$$

$$\text{and} \quad \left(\frac{\rho'}{\tau\rho^2} - \frac{\tau'}{\rho\tau^2}\right)t + \frac{1}{\rho} = 0$$

The coordinates of the edge of regression of the rectifying developable are

$$= x + \frac{\alpha\rho - \lambda\tau}{\tau'\rho - \rho'\tau}\tau, \quad \eta = y + \frac{\beta\rho - \mu\tau}{\tau'\rho - \rho'\tau}\tau, \quad \zeta = z + \frac{\gamma\rho - \nu\tau}{\tau'\rho - \rho'\tau}\tau$$

1. Under what conditions does the equation $F(x, y, z) = 0$ define a developable?

Assume that x, y, z are functions of two parameters u, v , such that the curves $v = \text{const}$ are the generators, and $u = \text{const}$ are any other lines. The equation of the plane is

$$(X-x)\frac{\partial F}{\partial x} + (Y-y)\frac{\partial F}{\partial y} + (Z-z)\frac{\partial F}{\partial z} = 0$$

It should involve u and be independent of v . Its characteristic is (1) and

$$A\frac{\partial x}{\partial u} + B\frac{\partial y}{\partial u} + C\frac{\partial z}{\partial u} = 0,$$

we have put, for the sake of brevity,

$$A = (X-x)\frac{\partial^2 F}{\partial x^2} + (Y-y)\frac{\partial^2 F}{\partial x \partial y} + (Z-z)\frac{\partial^2 F}{\partial x \partial z} - \frac{\partial F}{\partial x},$$

$$B = (X-x)\frac{\partial^2 F}{\partial x \partial y} + (Y-y)\frac{\partial^2 F}{\partial y^2} + (Z-z)\frac{\partial^2 F}{\partial y \partial z} - \frac{\partial F}{\partial y},$$

$$C = (X-x)\frac{\partial^2 F}{\partial x \partial z} + (Y-y)\frac{\partial^2 F}{\partial y \partial z} + (Z-z)\frac{\partial^2 F}{\partial z^2} - \frac{\partial F}{\partial z}$$

Since (1) is independent of v , we have

$$A\frac{\partial x}{\partial v} + B\frac{\partial y}{\partial v} + C\frac{\partial z}{\partial v} = 0$$

By equations (ii) and (iii) with (13), we see that

$$A - \lambda\frac{\partial F}{\partial x} = 0, \quad B - \lambda\frac{\partial F}{\partial y} = 0, \quad C - \lambda\frac{\partial F}{\partial z} = 0,$$

where λ denotes a factor of proportionality. If we eliminate $X - x$, $Y - y$, Z and λ from these equations and (1), we obtain the desired condition

$$\begin{vmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x \partial z} & \frac{\partial F}{\partial x} \\ \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial z} & \frac{\partial F}{\partial y} \\ \frac{\partial^2 F}{\partial x \partial z} & \frac{\partial^2 F}{\partial y \partial z} & \frac{\partial^2 F}{\partial z^2} & \frac{\partial F}{\partial z} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} & 0 \end{vmatrix} = 0$$

EXAMPLES

1 Find the envelope and edge of regression of the family of planes normal to a given curve

2 Find the rectifying developable of a cylindrical helix

3 Prove that the rectifying developable of a curve is the polar developable of its involutes, and conversely

4 Find the edge of regression of the envelope of the planes

$$x \sin u - y \cos u + z - au = 0$$

5 Determine the envelope of a one-parameter family of planes parallel to a given line

6 Given a one-parameter family of planes which cut the xy plane under a constant angle, the intersections of these planes with the latter plane envelope a curve C . Show that the edge of regression of the envelope of the planes is the evolute of C .

7 When a plane curve lies on a developable surface its plane meets the tangent planes to the surface in the tangent lines to the curve. Determine the developable surface which passes through a parabola and the circle, described in a perpendicular plane, on the latus rectum for diameter, and show that it is a cone.

8 Determine the developable surface which passes through the two parabolas $y^2 = 4ax$, $z = 0$, $x^2 = 4ay$, $z = b$, and show that its edge of regression lies on the surface $y^2z = x^3(b - z)$.

28 Applications of the moving trihedral Problems concerning the envelope of a family of surfaces are sometimes more readily solved when the surfaces are referred to the moving trihedral of a curve, which is associated in some manner with the family of surfaces, the parameter of points on the curve being the parameter of the family.

Let

(30)

$$F(\xi, \eta, \zeta, s) = 0$$

each a family of surfaces. Since ξ, η, ζ are functions of s , the equations of the characteristics are (30) and

$$\frac{dF}{ds} = \frac{\partial F}{\partial \xi} \frac{d\xi}{ds} + \frac{\partial F}{\partial \eta} \frac{d\eta}{ds} + \frac{\partial F}{\partial \zeta} \frac{d\zeta}{ds} + \frac{\partial F}{\partial s} = 0$$

characteristics being fixed in space, we have (I, 84)

$$\frac{d\xi}{ds} = \frac{\eta}{\rho} - 1, \quad \frac{d\eta}{ds} = -\left(\frac{\xi}{\rho} + \frac{\zeta}{\tau}\right), \quad \frac{d\zeta}{ds} = \frac{\eta}{\tau}$$

the equations of the characteristics are

$$F = 0, \quad \frac{\partial F}{\partial \xi} \left(\frac{\eta}{\rho} - 1\right) - \frac{\partial F}{\partial \eta} \left(\frac{\xi}{\rho} + \frac{\zeta}{\tau}\right) + \frac{\partial F}{\partial \zeta} \frac{\eta}{\tau} + \frac{\partial F}{\partial s} = 0$$

for the sake of brevity, we let $\phi(\xi, \eta, \zeta, s) = 0$ denote the second equation, the edge of regression is defined by (32) and

$$\frac{\partial \phi}{\partial \xi} \left(\frac{\eta}{\rho} - 1\right) - \frac{\partial \phi}{\partial \eta} \left(\frac{\xi}{\rho} + \frac{\zeta}{\tau}\right) + \frac{\partial \phi}{\partial \zeta} \frac{\eta}{\tau} + \frac{\partial \phi}{\partial s} = 0$$

For example, the family of osculating planes of a curve is defined with reference to the moving trihedral by $\zeta = 0$. In this case the second of (32) is $\eta = 0$, and $\frac{\xi}{\rho} = 0$. Hence the tangents are the characteristics, and the edge of regression of the curve, for, we have $\xi = \eta = \zeta = 0$.

In a similar manner the family of normal planes is defined by $\xi = 0$. Now the second of (32) is $-\rho = 0$, consequently the polar lines are the characteristics. Equation (33) becomes $\zeta + \rho \tau = 0$, hence the locus of the centers of the osculating spheres is the edge of regression (cf. § 18). The envelope is called the *polar developable*.

The osculating spheres of a twisted curve constitute a family of surfaces which is readily studied by the foregoing methods. From (18) it follows that the equation of these spheres is

$$\xi^2 + \eta^2 + \zeta^2 - 2\rho\eta + 2\rho'\tau\zeta = 0$$

and of equations (32) for this case is

$$\zeta \left[\frac{\rho}{\tau} + (\tau\rho')' \right] = 0,$$

since spherical curves are not considered, reduces to $\zeta = 0$. Equation (33) is $\eta = 0$, so that the coordinates of the edge of regression are $\xi = \eta = \zeta = 0$. Hence

osculating circles of a curve are the characteristics of its osculating spheres, and the curve itself is the edge of regression of the envelope of the spheres

29 Envelope of spheres Canal surfaces We consider now any family of spheres of one parameter. Referred to the moving trihedron of the curve of centers, the equation of the spheres is

$$\xi^2 + \eta^2 + \zeta^2 - r^2 = 0$$

By means of (32) we find that a characteristic is the circle in which a sphere is cut by the plane

$$\xi + rr' = 0$$

The radius of this circle is equal to $r\sqrt{1-r'^2}$. Hence the characteristic is imaginary when $r'^2 > 1$, reduces to a point when $r'^2 = 1$, and is real for $r'^2 < 1$.

By means of (33) we find that the coordinates of the edge of regression are given by

$$(34) \quad \xi = -rr', \quad \eta = [1 - (rr')']\rho, \quad \zeta^2 = r^2(1 - r'^2) - \rho^2[1 - (rr')']^2$$

Hence the edge of regression consists of two parts with corresponding points symmetrically placed with respect to the osculating plane of the curve of centers C , unless

$$r^2(1 - r'^2) - \rho^2[1 - (rr')']^2 = 0$$

When this condition is satisfied the edge is a single curve, and its points lie in the osculating planes of C . We have seen that this is the case with the osculating spheres of a curve. We shall show that when the above condition is satisfied the spheres osculate their edge of regression C_1 .

We write the above equation in the form

$$(35) \quad \rho[1 - (rr')'] = er\sqrt{1 - r'^2},$$

where e is $+1$ or -1 , so that ρ may be positive.

We have seen (§ 16) that the absolute and relative rates of change with s of the coordinates ξ, η, ζ of a point on C_1 are in the relation

$$\frac{\delta\xi}{\delta s} = \frac{d\xi}{ds} - \frac{\eta}{\rho} + 1, \quad \frac{\delta\eta}{\delta s} = \frac{d\eta}{ds} + \frac{\xi}{\rho} + \frac{\zeta}{\tau}, \quad \frac{\delta\zeta}{\delta s} = \frac{d\zeta}{ds} - \frac{\eta}{\tau}$$

When the values (34) are substituted in the right hand member of these equations, we obtain, in consequence of (35),

$$\frac{\delta\xi}{\delta s} = 0, \quad \frac{\delta\eta}{\delta s} = 0, \quad \frac{\delta\zeta}{\delta s} = -\frac{er\sqrt{1 - r'^2}}{\tau}$$

ENVELOPE OF SPHERES

linear element δs_1 of C_1 is given by

$$\delta s_1 = \frac{er\sqrt{1-r'^2}}{\tau} ds,$$

$$\frac{\delta \xi}{\delta s_1} = 0, \quad \frac{\delta \eta}{\delta s_1} = 0, \quad \frac{\delta \zeta}{\delta s_1} = -1$$

are the direction-cosines of the tangent to C_1 , we see that the tangent is normal to the osculating plane to the curve of C_1 . Moreover, these direction cosines must satisfy (cf I, 83) the relations

$$= \frac{da}{ds} - \frac{b}{\rho}, \quad \frac{\delta b}{\delta s} = \frac{db}{ds} + \frac{a}{\rho} + \frac{c}{\tau}, \quad \frac{\delta c}{\delta s} = \frac{dc}{ds} - \frac{b}{\tau}$$

have

$$\frac{\delta^2 \xi}{\delta s_1^2} = 0, \quad \frac{\delta^2 \eta}{\delta s_1^2} = -\frac{1}{er\sqrt{1-r'^2}}, \quad \frac{\delta^2 \zeta}{\delta s_1^2} = 0,$$

hence it follows that the radius of curvature ρ_1 of C_1 is

$$\rho_1 = ee'r\sqrt{1-r'^2},$$

either +1 or -1, so that ρ_1 may be positive or negative. Since, now, the direction cosines of the principal normal have the values

$$\rho_1 \frac{\delta^2 \xi}{\delta s_1^2} = 0, \quad \rho_1 \frac{\delta^2 \eta}{\delta s_1^2} = -e', \quad \rho_1 \frac{\delta^2 \zeta}{\delta s_1^2} = 0,$$

that the principal normals to C and C_1 are parallel. Therefore, since these quantities must satisfy equations (37),

$$\frac{\delta^3 \xi}{\delta s_1^3} = \frac{\tau}{\rho \rho_1^2}, \quad \frac{\delta^3 \eta}{\delta s_1^3} = \frac{e'}{\rho_1^2} \rho_1', \quad \frac{\delta^3 \zeta}{\delta s_1^3} = -\frac{1}{\rho_1^2},$$

denotes the derivative of ρ_1 with respect to s_1 . By means of these relations we find that the radius of torsion τ_1 of C_1 is given by

(38) we find $\rho_1' = \frac{rr'\tau}{\rho\rho'}$, so that the radius R_1 of the osculating sphere of C_1 is given by $R_1^2 = \rho_1^2 + \rho_1'^2 \tau_1^2 = r^2$, and consequently the osculating spheres of C_1 are of the same radius as the given

The direction cosines of the tangent, principal normal, and binormal to C_1 are found from (36) and (39) to be

$$\alpha_1 = \beta_1 = 0, \quad \gamma_1 = -1, \quad l_1 = n_1 = 0, \quad m_1 = -e', \quad \lambda_1 = -e', \quad \mu_1 = \nu_1 =$$

Hence the coordinates (I, 94) of the center of the osculating sphere of C_1 are reducible, in consequence of (34), to

$$\xi + l_1 \rho_1 - \rho'_1 \tau_1 \lambda_1 = 0, \quad \eta + m_1 \rho_1 - \rho'_1 \tau_1 \mu_1 = 0, \quad \zeta + n_1 \rho_1 - \rho'_1 \tau_1 \nu_1 = 0$$

Therefore we have the theorem

When the edge of regression of a family of spheres of one parameter has only one branch, the spheres osculate the edge

Since τ does not appear in equation (35), it follows that when r is given as a function of s , the intrinsic equations of the curve centers are

$$\rho = \frac{er\sqrt{1-r'^2}}{1-(rr')'}, \quad \tau = f(s),$$

where the function $f(s)$ is arbitrary. Moreover, any curve will serve for the curve of centers of such an envelope of spheres. The determination of r requires the solution of equation (35) and consequently involves two arbitrary constants.

When all the spheres of a family have the same radius, the envelope is called a *canal surface*. From (34) it is seen that in this case a characteristic is a great circle. Moreover, equation (35) reduces to $\rho = r$. Hence a necessary and sufficient condition that the edge of regression of a canal surface consist of a single curve is that the curve of centers be of constant curvature and the radius of the sphere equal to the radius of first curvature of the curve.

GENERAL EXAMPLES

1. Let MN be a generator of the right conoid

$$x = u \cos v, \quad y = u \sin v, \quad z = 2k \operatorname{cosec} 2v,$$

M being the point in which it meets the z axis. Show that the tangent plane to the surface in a hyperbola which passes through M , and that as N moves along the generator the tangent at M to the hyperbola describes a plane.

2. A point moves on an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, so that the direction of motion always passes through the perpendicular from the center on the tangent plane at the point. Show that the path of the point is the curve in which the ellipsoid is cut by the surface $xy^m z^n = \text{const}$, where $\frac{1}{l} = \frac{1}{b^2} - \frac{1}{c^2}$, $\frac{1}{m} = \frac{1}{c^2} - \frac{1}{a^2}$, $\frac{1}{n} = \frac{1}{a^2} - \frac{1}{b^2}$.

each of the generators of a developable surface be revolved through the same angle about the tangent to an orthogonal trajectory of the generators at the intersection, the locus of these lines is a developable surface whose edge of regression is an evolute of the given trajectory

Show that the edge of regression of the family of planes

$$(1 - u^2)x + 2(1 + u^2)y + 2uz + f(u) = 0$$

is a curve

Find the developable surface which passes through the circles $x^2 + y^2 = a^2$, $z = 0$, b^2 , $y = 0$ meets the plane $x = 0$ in an equilateral hyperbola.

Find the edge of regression of the developable surface which envelopes the cylinder $x^2 + y^2 = by$ along the curve in which the latter is cut by the cylinder $x^2 = by$

Find the envelope of the planes which pass through the center of an ellipsoid in sections of equal area

Find the first and second curvatures of the edge of regression of the family of planes $\alpha x + \beta y + \gamma z = p$, where α , β , γ , p are functions of a single parameter u and $\alpha^2 + \beta^2 + \gamma^2 = 1$, are given by

$$\frac{1}{\rho} = \frac{\Delta^2}{D(\alpha'^2 + \beta'^2 + \gamma'^2)^{\frac{3}{2}}}, \quad \frac{1}{\tau} = \frac{\Delta^2}{D},$$

$$\Delta = \begin{vmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{vmatrix}, \quad D = \begin{vmatrix} p & p' & p'' \\ \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{vmatrix}$$

Derive the equations of the edge of regression of the rectifying developable of a curve

Derive the results of § 29 without the aid of the moving trihedral

Find the envelope of the spheres whose diameters are the chords of a circle and whose centers are the points of the latter

Find the envelope and edge of regression of the spheres which pass through a fixed point and whose centers lie on a given curve

Find the envelope and edge of regression of the spheres which have for their diameters the chords of a family of circular sections of an ellipsoid

Find the envelope and edge of regression of the family of ellipsoids $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where a is the parameter

Find the envelope of the family of spheres whose diameters are parallel to a fixed line and whose centers lie on an ellipse

Find the equations of the canal surface whose curve of centers is a circular helix and whose edge of regression has one branch. Determine the latter

Find the envelope of the family of cones

$$(ax + x + y + z - 1)(ay + z) - ax(x + y + z - 1) = 0,$$

where a is the parameter

CHAPTER III

LINEAR ELEMENT OF A SURFACE DIFFERENTIAL PARAMETERS CONFORMAL REPRESENTATION

30 Linear element Upon a surface S , defined by equations in the parametric form

$$(1) \quad x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v),$$

we select any curve and write its equations $\phi(u, v) = 0$. From § 3 we have that the linear element of the curve is given by

$$(2) \quad ds^2 = dx^2 + dy^2 + dz^2,$$

where $dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$, $dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$, $dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv$,

the differentials du, dv satisfying the condition

$$\frac{\partial \phi}{\partial u} du + \frac{\partial \phi}{\partial v} dv = 0$$

If we put

$$(3) \quad \begin{cases} E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2, \\ F = \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v}, \\ G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2, \end{cases}$$

or, in abbreviated form,

$$E = \sum \left(\frac{\partial x}{\partial u}\right)^2, \quad F = \sum \frac{\partial x}{\partial u} \frac{\partial x}{\partial v}, \quad G = \sum \left(\frac{\partial x}{\partial v}\right)^2,$$

equation (2) becomes

$$(4) \quad ds^2 = E du^2 + 2F du dv + G dv^2$$

The functions E, F, G thus defined were first used by Gauss *. When the surface is real, and likewise the curvilinear coordinates

* *Disquisitiones generales circa superficies curvas* (English translation by Morehead and Hiltebeitel), p 18. Princeton, 1902. Unless otherwise stated, all references to Gauss are to this translation.

functions $\sqrt{E}$, $\sqrt{G}$ are real. We shall understand also that E and G are positive. There is, however, an important exception, namely when both E and G are zero (cf § 35).

Any other curve equation (4) will have the same form, but the relation between du and dv will depend upon the curve. The value of ds , given by (4), is the element of arc length on the surface. It is called the *linear element of the surface* (cf § 20). However, in order to avoid circumlocution, we frequently call the expression for ds^2 the *linear element*, or the *fundamental quadratic form*. The coefficients of the last, E , F , G , are called the *fundamental quantities of the surface*.

For the sake of brevity, we put

$$A = \frac{\partial(y, z)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}, \quad B = \frac{\partial(z, x)}{\partial(u, v)}, \quad C = \frac{\partial(x, y)}{\partial(u, v)},$$

from (3) and (5) that

$$EG - F^2 = A^2 + B^2 + C^2$$

when the surface is real and likewise the parameters, $EG - F^2$ is different from zero unless A , B , and C are all zero. But if A , B , and C are zero, it follows from (5) that u and v are not independent, and consequently equations (1) define a curve and not a surface. However, it may happen that for certain values of u and v all the quantities A , B , C vanish. Such points are called *singular points* of the surface. These points may be isolated or constitute one or more curves on the surface, such curves are called *singular lines*. In the following discussion only ordinary points will be considered.

From the preceding remarks it follows that for real surfaces, and for real coordinate lines, the function H defined by

$$H = \sqrt{EG - F^2}$$

is always positive by hypothesis.

LINEAR ELEMENT OF A SURFACE

Isotropic developable The exceptional case, where the surface is imaginary, is afforded by the tangent surface of a minimal curve. The equations of such a surface are (cf § 22)

$$\begin{aligned}x &= \int \frac{1-u^2}{2} \phi(u) du + \frac{1-u^2}{2} \phi(u) v, \\y &= \int \frac{1+u^2}{2} \phi(u) du + \frac{1+u^2}{2} \phi(u) v, \\z &= \int u \phi(u) du + u \phi(u) v,\end{aligned}$$

where $\phi(u)$ is a function of u different from zero. It is readily found that $E = v^2 \phi^2(u)$, $F = G = 0$, and consequently $EG - F^2 = 0$. This equation is likewise the sufficient condition that the surface be of the kind sought. For, when it is satisfied, the equation of the linear element can be written $ds^2 = (\sqrt{E} du + \sqrt{G} dv)^2$. If λ denotes an integrating factor of $\sqrt{E} du + \sqrt{G} dv$, and a function u_1 be defined by the equation $\lambda(\sqrt{E} du + \sqrt{G} dv) = du_1$, the above equation becomes $ds^2 = \frac{1}{\lambda^2} du_1^2$. Hence, if we take for parametric curves $u_1 = \text{const}$ and any other system for $v_1 = \text{const}$, we have $F_1 = 0$, $G_1 = 0$. In other form these equations are

$$\frac{\partial x}{\partial u_1} \frac{\partial x}{\partial v_1} + \frac{\partial y}{\partial u_1} \frac{\partial y}{\partial v_1} + \frac{\partial z}{\partial u_1} \frac{\partial z}{\partial v_1} = 0, \quad \left(\frac{\partial x}{\partial v_1} \right)^2 + \left(\frac{\partial y}{\partial v_1} \right)^2 + \left(\frac{\partial z}{\partial v_1} \right)^2 = 0$$

In accordance with the last equation we put

$$\frac{\partial x}{\partial v_1} = k \frac{1-u_1^2}{2}, \quad \frac{\partial y}{\partial v_1} = ik \frac{1+u_1^2}{2}, \quad \frac{\partial z}{\partial v_1} = ku_1,$$

where k is undetermined.

By integration we have

$$(1) \quad x = \frac{1-u_1^2}{2} \int k dv_1 + \lambda, \quad y = i \frac{1+u_1^2}{2} \int k dv_1 + \mu, \quad z = u_1 \int k dv_1 + \nu,$$

λ , μ , ν being functions of u_1 alone. When these values are substituted in the first of the above equations of condition, we get

$$(1-u_1^2) \frac{\partial \lambda}{\partial u_1} + i(1+u_1^2) \frac{\partial \mu}{\partial u_1} + 2u_1 \frac{\partial \nu}{\partial u_1} = 0,$$

to be satisfied by λ , μ , and ν .

The equation of the tangent plane to the surface (1) is reducible to

$$(1-u_1^2)(X-x) + i(1+u_1^2)(Y-y) + 2u_1(Z-z) = 0$$

Hence the surface is developable. Since its edge of regression is a minimal curve (Ex 4, p 69), the theorem is proved. The surface is called an *isotropic developable*.

32 Transformation of coordinates It is readily found that the functions E , F , G are unaltered in value by any change of the rectangular axes. But now we shall show that these functions change their values when there is a change of the curvilinear coordinates.

the transformation of coordinates be defined by the equations

$$u = u(u_1, v_1), \quad v = v(u_1, v_1),$$

have

$$\frac{\partial x}{\partial u_1} = \frac{\partial x}{\partial u} \frac{\partial u}{\partial u_1} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial u_1}, \quad \frac{\partial x}{\partial v_1} = \frac{\partial x}{\partial u} \frac{\partial u}{\partial v_1} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial v_1}$$

$$\text{let } E_1 = \sum \left(\frac{\partial x}{\partial u_1} \right)^2, \quad F_1 = \sum \frac{\partial x}{\partial u_1} \frac{\partial x}{\partial v_1}, \quad G_1 = \sum \left(\frac{\partial x}{\partial v_1} \right)^2,$$

the relations

$$\begin{cases} E_1 = E \left(\frac{\partial u}{\partial u_1} \right)^2 + 2F \frac{\partial u}{\partial u_1} \frac{\partial v}{\partial u_1} + G \left(\frac{\partial v}{\partial u_1} \right)^2, \\ F_1 = E \frac{\partial u}{\partial u_1} \frac{\partial u}{\partial v_1} + F \left(\frac{\partial u}{\partial u_1} \frac{\partial v}{\partial v_1} + \frac{\partial u}{\partial v_1} \frac{\partial v}{\partial u_1} \right) + G \frac{\partial v}{\partial u_1} \frac{\partial v}{\partial v_1}, \\ G_1 = E \left(\frac{\partial u}{\partial v_1} \right)^2 + 2F \frac{\partial u}{\partial v_1} \frac{\partial v}{\partial v_1} + G \left(\frac{\partial v}{\partial v_1} \right)^2 \end{cases}$$

the fundamental quantities of the first order assume new values when there is a change of curvilinear coordinates

(8) we have, by differentiation,

$$du = \frac{\partial u}{\partial u_1} du_1 + \frac{\partial u}{\partial v_1} dv_1, \quad dv = \frac{\partial v}{\partial u_1} du_1 + \frac{\partial v}{\partial v_1} dv_1$$

these equations for du_1, dv_1 , we get

$$= \frac{1}{\delta} \left(\frac{\partial v}{\partial v_1} du - \frac{\partial u}{\partial v_1} dv \right), \quad dv_1 = \frac{1}{\delta} \left(-\frac{\partial v}{\partial u_1} du + \frac{\partial u}{\partial u_1} dv \right),$$

$$\delta = \frac{\partial(u, v)}{\partial(u_1, v_1)}$$

we have

$$\begin{cases} \frac{\partial u_1}{\partial u} = \frac{1}{\delta} \frac{\partial v}{\partial v_1}, & \frac{\partial u_1}{\partial v} = -\frac{1}{\delta} \frac{\partial u}{\partial v_1}, \\ \frac{\partial v_1}{\partial u} = -\frac{1}{\delta} \frac{\partial v}{\partial u_1}, & \frac{\partial v_1}{\partial v} = \frac{1}{\delta} \frac{\partial u}{\partial u_1}, \end{cases}$$

$$\frac{\partial(u_1, v_1)}{\partial(u, v)} = \frac{1}{\delta}$$

(9) we find the relation

$$E_1 G_1 - F_1^2 = \delta^2 (EG - F^2)$$

By means of this equation and the relations (11), we can transform equations (9) into the following

$$(13) \quad \begin{cases} \frac{E_1}{E_1 G_1 - F_1^2} = \frac{E \left(\frac{\partial v_1}{\partial v} \right)^2 - 2F \frac{\partial v_1}{\partial u} \frac{\partial v_1}{\partial v} + G \left(\frac{\partial v_1}{\partial u} \right)^2}{EG - F^2}, \\ \frac{-F_1}{E_1 G_1 - F_1^2} = \frac{E \frac{\partial u_1}{\partial v} \frac{\partial v_1}{\partial v} - F \left(\frac{\partial u_1}{\partial u} \frac{\partial v_1}{\partial v} + \frac{\partial u_1}{\partial v} \frac{\partial v_1}{\partial u} \right) + G \frac{\partial u_1}{\partial u} \frac{\partial v_1}{\partial u}}{EG - F^2}, \\ \frac{G_1}{E_1 G_1 - F_1^2} = \frac{E \left(\frac{\partial u_1}{\partial v} \right)^2 - 2F \frac{\partial u_1}{\partial u} \frac{\partial u_1}{\partial v} + G \left(\frac{\partial u_1}{\partial u} \right)^2}{EG - F^2} \end{cases}$$

33 Angles between curves The element of area Upon a parametric line $v = \text{const}$ we take for positive sense the direction in which the parameter u increases, and likewise upon a curve $u = \text{const}$ the direction in which v increases. If ds_v and ds_u denote the elements of arc of curves $v = \text{const}$ and $u = \text{const}$ respectively, we find, from (4),

$$(14) \quad ds_v = \sqrt{E} du, \quad ds_u = \sqrt{G} dv$$

Hence, if $\alpha_v, \beta_v, \gamma_v$ and $\alpha_u, \beta_u, \gamma_u$ denote the direction-cosines of the tangents to these curves respectively, we have

$$\alpha_v = \frac{1}{\sqrt{E}} \frac{\partial x}{\partial u}, \quad \beta_v = \frac{1}{\sqrt{E}} \frac{\partial y}{\partial u}, \quad \gamma_v = \frac{1}{\sqrt{E}} \frac{\partial z}{\partial u},$$

$$\alpha_u = \frac{1}{\sqrt{G}} \frac{\partial x}{\partial v}, \quad \beta_u = \frac{1}{\sqrt{G}} \frac{\partial y}{\partial v}, \quad \gamma_u = \frac{1}{\sqrt{G}} \frac{\partial z}{\partial v}$$

We have seen that through an ordinary point of a surface there passes one curve of parameter u and one of parameter v . If, as in fig 11, ω denotes the angle, between 0° and 180° , formed by the positive directions of the tangents to these curves at the point, we have

$$(15) \quad \cos \omega = \alpha_u \alpha_v + \beta_u \beta_v + \gamma_u \gamma_v = \frac{F}{\sqrt{EG}},$$

and

$$(16) \quad \sin \omega = \frac{\sqrt{EG - F^2}}{\sqrt{EG}} = \frac{H}{\sqrt{EG}}$$

two families of curves upon a surface are such that any point a curve of each family, and but one, passes, moreover, the tangents at a point to the two curves it are perpendicular, the curves are said to form an *orthogonal system*. From (15) we have the theorem

Necessary and sufficient condition that the parametric lines upon a surface form an orthogonal system is that $F=0$

Consider the small quadrilateral (fig 11) whose vertices are (u, v) , $(u+du, v)$, $(u+du, v+dv)$, $(u, v+dv)$. To within terms of higher order the sides of the figure are equal. Consequently it is approxi-

parallel to a parallelogram whose sides are of length $\sqrt{E} du$ and $\sqrt{G} dv$ and the included angle is ω . The area of this parallelogram is called the *elementary area of the surface*. Its value is

$$dA = \sin \omega \sqrt{EG} du dv = H du dv$$

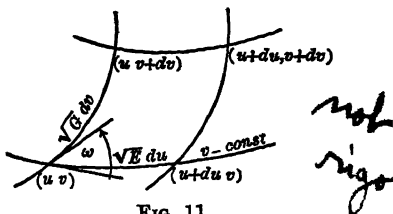


FIG 11

Let any curve on a surface, the direction-cosines α , β , γ of the tangent at a point have the form

$$\begin{aligned} \frac{dx}{ds} &= \left(\frac{\partial x}{\partial u} \frac{du}{ds} + \frac{\partial x}{\partial v} \frac{dv}{ds} \right), & \beta &= \frac{dy}{ds} = \left(\frac{\partial y}{\partial u} \frac{du}{ds} + \frac{\partial y}{\partial v} \frac{dv}{ds} \right), \\ \gamma &= \frac{dz}{ds} = \left(\frac{\partial z}{\partial u} \frac{du}{ds} + \frac{\partial z}{\partial v} \frac{dv}{ds} \right) \end{aligned}$$

If $dv/du = \lambda$ and replace ds by the positive square root of the right-hand member of (4), the above expressions can be written

$$\begin{cases} \alpha = \frac{\frac{\partial x}{\partial u} + \lambda \frac{\partial x}{\partial v}}{\sqrt{E + 2F\lambda + G\lambda^2}}, \\ \beta = \frac{\frac{\partial y}{\partial u} + \lambda \frac{\partial y}{\partial v}}{\sqrt{E + 2F\lambda + G\lambda^2}}, \\ \gamma = \frac{\frac{\partial z}{\partial u} + \lambda \frac{\partial z}{\partial v}}{\sqrt{E + 2F\lambda + G\lambda^2}} \end{cases}$$

From these results it is seen that the direction-cosines depend not upon the absolute values of du and dv , but upon their ratio λ . The value of λ is obtained by differentiation from the equation of C , namely

$$(19) \quad \phi(u, v) = 0$$

Let C_1 be a second curve meeting C at a point M , and let the direction cosines of the tangent to C_1 at M be $\alpha_1, \beta_1, \gamma_1$. They are given by

$$(20) \quad \alpha_1 = \frac{\partial x}{\partial u} \frac{\delta u}{\delta s} + \frac{\partial x}{\partial v} \frac{\delta v}{\delta s},$$

and similar expressions for β_1 and γ_1 , where δ indicates variation in the direction of C_1 .

If θ denotes the angle between the positive directions to C and C_1 at M , we have, from (18) and (20),

$$(21) \quad \cos \theta = \alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1 = \frac{E du \delta u + F(du \delta v + dv \delta u) + G dv \delta v}{ds \delta s},$$

and

$$\sin \theta = \pm \sqrt{1 - \cos^2 \theta} = \pm H \left(\frac{\delta u}{\delta s} \frac{dv}{ds} - \frac{\delta v}{\delta s} \frac{du}{ds} \right)$$

This ambiguity of sign is due to the fact that θ as defined is one of two angles which together are equal to 360° . We take the upper sign, thus determining θ . This gives

$$(22) \quad \sin \theta = H \left(\frac{\delta u}{\delta s} \frac{dv}{ds} - \frac{\delta v}{\delta s} \frac{du}{ds} \right)$$

The significance of the above choice will be pointed out shortly.

When in particular C_1 is the curve $v = \text{const}$ through M , we have $\delta v = 0$ and $\delta s = \sqrt{E} \delta u$, so that

$$(23) \quad \cos \theta_0 = \frac{1}{\sqrt{E}} \left(E \frac{du}{ds} + F \frac{dv}{ds} \right), \quad \sin \theta_0 = \frac{H}{\sqrt{E}} \frac{dv}{ds}$$

From these equations we obtain

$$(24) \quad \tan \theta_0 = \frac{H dv}{E du + F dv}$$

The angle ω between the positive half tangents to the parametric curves has been uniquely defined. Hence there is, in

only one sense in which the tangent to the curve $v = \text{const.}$ is brought into coincidence with the tangent to the curve $u = \text{const.}$ by a rotation of amount ω . We say that rotations in this direction are *positive*, in the opposite sense *negative*. From this it is seen that θ_0 is the angle described in the positive sense when a positive half tangent to the curve $v = \text{const.}$ is rotated into coincidence with the half tangent to C . And so in the general case defined by (22), ω is the angle described in the positive rotation of the second curve to the first.

From equations (15), (16), and (23) we find

$$\sin(\omega - \theta_0) = \frac{1}{\sqrt{G}} \left(F \frac{du}{ds} + G \frac{dv}{ds} \right), \quad \sin(\omega - \theta_0) = \frac{H}{\sqrt{G}} \frac{du}{ds}$$

Equations follow also directly from (20) and (21) by considering the curve $u = \text{const.}$ as the second line.

As an immediate consequence of equation (21) we have the

Necessary and sufficient condition that the tangents to two curves on a surface at a point of meeting be perpendicular is

$$E du \delta u + F (du \delta v + dv \delta u) + G dv \delta v = 0$$

EXAMPLES

1. Show that when the equation of a surface is of the form $z = f(x, y)$, its linear element can be written

$$ds^2 = (1 + p^2) dx^2 + 2pq dx dy + (1 + q^2) dy^2,$$

where $p = \partial z / \partial x$, and $q = \partial z / \partial y$. Under what conditions do the lines $x = \text{const.}$ and $y = \text{const.}$ form an orthogonal system?

2. Show that the parametric curves on the sphere

$$x = a \sin u \cos v, \quad y = a \sin u \sin v, \quad z = a \cos u$$

form an orthogonal system. Determine the two families of curves which meet the sphere at constant angles $\pi/4$ and $3\pi/4$. Find the linear element of the sphere in these new curves as parametric.

3. Find the equation of a curve on the paraboloid of revolution $z = u \cos v$, $x = u^2/2$, which meets the curves $v = \text{const.}$ under constant angle α through two points (u_0, v_0) , (u_1, v_1) . Determine α as a function of v_1 .

4. Find the differential equation of the curves upon the tangent surface of a cylinder which cut the generators under constant angle α .

5 Show that the equations of a curve which lies upon a right cone and cuts all the generators under the same angle are of the form $x = ce^{au} \cos u$, $y = ce^{au} \sin u$, $z = be^{au}$, where a , b , and c are constants. What is the projection of the curve upon a plane perpendicular to the axis of the cone? Find the radius of curvature of the curve.

6 Find the equations of the curves which bisect the angles between the parametric curves of the paraboloid in Ex. 3.

34 Families of curves An equation of the form

$$(27) \quad \phi(u, v) = c,$$

where c is an arbitrary constant, defines an infinity of curves, or a *family of curves*, upon the surface. Through any point of the surface there passes a curve of the family. For, given the curvilinear coordinates of a point, when these values are substituted in (27) we obtain a value of c , say c_0 , then evidently the curve $\phi = c_0$ passes through the point. We inquire whether this family of curves can be defined by another equation. Suppose it is possible, and let the equation be

$$(28) \quad \psi(u, v) = \kappa$$

Since c and κ are constant along any curve and vary in passing from one curve to another, each is necessarily a function of the other. Hence ψ is a function of ϕ . Moreover, if ψ is any function of ϕ , equations (27) and (28) define the same family of curves.

From equation (24) it is seen that the direction, at any point, of the curve of the family through the point is determined by the value of dv/du . We obtain the latter from the equation

$$(29) \quad \frac{\partial \phi}{\partial u} du + \frac{\partial \phi}{\partial v} dv = 0,$$

which is derived from (27) by differentiation.

Let $\phi(u, v) = c$ be an integral of an ordinary differential equation of the first order and first degree, such as

$$(30) \quad M(u, v) du + N(u, v) dv = 0$$

The curves defined by the former equation are called *integral curves* of equation (30). From the integral equation we get equation (29) by differentiation. It must be possible then to obtain equation (30)

integral equation and (29) But c does not appear in consequently the latter equation differs from (30) by a factor

Hence $M \frac{\partial \phi}{\partial v} - N \frac{\partial \phi}{\partial u} = 0$ Suppose, now, that we have integral of (30), as $\psi(u, v) = c$ Then $M \frac{\partial \psi}{\partial v} - N \frac{\partial \psi}{\partial u} = 0$ Elimination of M and N from these equations gives $\frac{\partial(\phi, \psi)}{\partial(u, v)} = 0$;

which it follows that ψ is a function of ϕ Moreover, ψ can be a function of ϕ But we have seen that if ψ is a function of ϕ the families of curves $\phi = \text{const}$ and $\psi = \text{const}$ are the same All integrals of equation (30) of the form $\phi = c$ or $\psi = c$ are the same family of curves However, equation (30) may be an integral in which the constant of integration enters, as $F(u, v, c) = 0$ But if this be solved for c , we obtain more integrals of the form (27) Hence an equation of the form (30) defines one family of curves on a surface Although the determination of the curves when thus defined requires the solution of the equation, the direction of any curve at a point is determined directly by means of (24)

At each point of intersection of a curve C_1 with the curves of a family, the tangents to the two curves are perpendicular to each other, C_1 is called an *orthogonal trajectory of the curves* Suppose the family of curves is defined by equation (30) The direction of the tangents to the curves is determined by the ratios $\frac{dv}{du}$ and $\frac{\delta v}{\delta u}$, which determine the direction of the tangents to the two curves at the point of intersection, by equation (26) If we replace $\frac{\delta v}{\delta u}$ by $-\frac{M}{N}$, we obtain

$$(EN - FM) du + (FN - GM) dv = 0$$

The integral curve of this equation is an orthogonal trajectory of the family of curves Hence a family of curves admits of a family of orthogonal trajectories They are defined by equation (31), the differential equation of the curves is in the form (30) If the family is defined by a finite equation, such as (27), the determination of the orthogonal trajectories is

$$\left(E \frac{\partial \phi}{\partial v} - F \frac{\partial \phi}{\partial u}\right) du + \left(F \frac{\partial \phi}{\partial v} - G \frac{\partial \phi}{\partial u}\right) dv = 0$$

As an example, we consider the family of circles in the plane with centers on the x -axis whose equation is

$$(1) \quad x^2 + y^2 - 2ux = a^2,$$

where u is the parameter of the family and a is a constant. In order to find the orthogonal trajectories of these curves, we take the lines $x = \text{const}$, $y = \text{const}$ for parametric curves, in which case

$$E = G = 1, \quad F = 0,$$

and write the equation (1) in the form (27), thus

$$x + \frac{1}{x}(y^2 - a^2) = 2u$$

Now equation (32) is $2xy dx - (x^2 - y^2 + a^2) dy = 0$, of which the integral is

$$\frac{x^2 + a^2}{y} + y = 2v,$$

where v is the constant of integration. Hence the orthogonal trajectories are circles whose centers are on the y -axis.

An ordinary differential equation of the second degree, such as

$$(33) \quad R(u, v) du^2 + 2S(u, v) du dv + T(u, v) dv^2 = 0,$$

is equivalent to two equations of the first degree, which are found by solving this equation as a quadratic in dv . Hence equation (33) defines two families of curves upon the surface. We seek the condition that the curves of one family be the orthogonal trajectories of the other, or, in other words, the condition that (33) be the equation of an orthogonal system, as previously defined. If κ_1 and κ_2 denote the two values of $\frac{dv}{du}$ obtained from (33), we have

$$\kappa_1 + \kappa_2 = -\frac{2S}{T}, \quad \kappa_1 \kappa_2 = \frac{R}{T}.$$

From (26) it follows that the condition that the two directions at a point corresponding to κ_1 and κ_2 be perpendicular is

$$E + F(\kappa_1 + \kappa_2) + G\kappa_1\kappa_2 = 0$$

If the above values are substituted in this equation, we have the condition sought, it is

$$(34) \quad ET + GR - 2FS = 0$$

minimal curves on a surface. An equation of the form (33) is obtained by equating to zero the first fundamental form of the surface. This gives

$$E du^2 + 2F du dv + G dv^2 = 0,$$

which defines the double family of imaginary curves of length zero which lie on the surface. In this case equation (34) reduces to $r^2 = 0$, hence the minimal lines on a surface form an isotropic system only when the surface is an isotropic developable.

An important example of these lines is furnished by the system of great circles on a sphere. Here, if we take a sphere of unit radius and center at the origin, its equation, $x^2 + y^2 + z^2 = 1$, can be written in either of

$$\frac{x + iy}{1 - z} = \frac{1 + z}{x - iy} = u,$$

$$\frac{x - iy}{1 - z} = \frac{1 + z}{x + iy} = v,$$

and u and v denote the respective ratios, and evidently are conjugates. If these four equations are solved for x, y, z , we find

$$x = \frac{u + v}{uv + 1}, \quad y = \frac{i(v - u)}{uv + 1}, \quad z = \frac{uv - 1}{uv + 1}$$

In these expressions we find that the linear element, in terms of the parameters u and v , is given by

$$ds^2 = \frac{4 du dv}{(1 + uv)^2}$$

The curves $u = \text{const}$ and $v = \text{const}$ are the lines of length zero.

By eliminating u from the first two and the last two of equations (35) we get

$$\begin{cases} i(v^2 + 1)x + (1 - v^2)y - 2iv = 0, \\ i(v^2 + 1)z + 2vy + i(1 - v^2) = 0 \end{cases}$$

Thus the points of a curve $v = \text{const}$ lie on the line

$$\begin{aligned} i(v^2 + 1)X + (1 - v^2)Y - 2iv &= 0, \\ i(v^2 + 1)Z + 2vY + i(1 - v^2) &= 0, \end{aligned}$$

where X, Y, Z denote current coordinates. In consequence of (35), these equations can be written

$$\frac{X-x_0}{v^2-1} = \frac{Y-y_0}{v^2+1} = \frac{Z-z_0}{-2v},$$

y_0, z_0 are the coordinates of a particular point. In like manner, the curves $u = \text{const}$ are the minimal lines

$$\frac{X-x_0}{1-u^2} = \frac{Y-y_0}{1+u^2} = \frac{Z-z_0}{2u}$$

EXAMPLES

1 Show that the most general orthogonal system of circles in the plane is that of the example in § 34

2 Show that on the right conoid $x = u \cos v$, $y = u \sin v$, $z = av$, the curves $du^2 - (u^2 + a^2) dv^2 = 0$ form an orthogonal system

3 When the coefficients of the linear elements of two surfaces,

$$ds_1^2 = E_1 du^2 + 2F_1 dudv + G_1 dv^2, \quad ds_2^2 = E_2 du^2 + 2F_2 dudv + G_2 dv^2,$$

are not proportional, and points with the same curvilinear coordinates on each of the surfaces are said to correspond, there is a unique orthogonal system on one surface corresponding to an orthogonal system on the other, its equation is

$$(F_1 E_2 - F_2 E_1) du^2 + (E_2 G_1 - E_1 G_2) dudv + (G_1 F_2 - G_2 F_1) dv^2 = 0$$

4 If θ_1 and θ_2 are solutions of the equation

$$\frac{\partial^2 \theta}{\partial \alpha \partial \beta} - \frac{1}{2} \frac{\partial \log \lambda}{\partial \alpha} \frac{\partial \theta}{\partial \beta} - \lambda \theta = 0,$$

where λ is any function of α and β , the equations

$$x + y = \int \theta_1^2 d\alpha + \frac{1}{\lambda} \left(\frac{\partial \theta_1}{\partial \beta} \right)^2 d\beta,$$

$$x - y = \int \theta_2^2 d\alpha + \frac{1}{\lambda} \left(\frac{\partial \theta_2}{\partial \beta} \right)^2 d\beta,$$

$$z = \int \theta_1 \theta_2 d\alpha + \frac{1}{\lambda} \frac{\partial \theta_1}{\partial \beta} \frac{\partial \theta_2}{\partial \beta} d\beta,$$

define a surface referred to its minimal lines

36 Variation of a function Let S be a surface referred to any system of coordinates u, v , and let $\phi(u, v)$ be a function of u and v . When the values of the coordinates of a point M of the surface are substituted in ϕ , we obtain a number c , and consequently the curve

$$(38) \quad \phi(u, v) = c$$

rough M In a displacement from M along this curve the ϕ remains the same, but in any other direction it changes rate of change is given by

$$\frac{d\phi}{ds} = \frac{\frac{\partial\phi}{\partial u} + \frac{\partial\phi}{\partial v}k}{\sqrt{E + 2Fk + Gk^2}},$$

$= dv/du$ determines the direction As thus written it is good that the denominator of the right-hand member is

he present we consider the absolute value of $\frac{d\phi}{ds}$, and write

$$A = \left| \frac{d\phi}{ds} \right| = e \frac{\frac{\partial\phi}{\partial u} + \frac{\partial\phi}{\partial v}k}{\sqrt{E + 2Fk + Gk^2}},$$

is ± 1 according as the sign of the numerator is positive or $\pm$ The minimum value of A is zero and corresponds to the n along the curve (38) In order to find the maximum value te to zero the derivative of A with respect to k This gives

$$\left(E \frac{\partial\phi}{\partial v} - F \frac{\partial\phi}{\partial u} \right) + \left(F \frac{\partial\phi}{\partial v} - G \frac{\partial\phi}{\partial u} \right) k = 0$$

(2) it follows that this value of k determines the direction angles to the tangent to $\phi = c$ at the point By substituting ue of k in (39) we get the maximum value of A Hence

differential quotient $\frac{d\phi}{ds}$ of a function $\phi(u, v)$ at a point on a varies in value with the direction from the point It equals the direction tangent to the curve $\phi = c$, and attains its greatest value in the direction normal to this curve, this value being

$$\left| \frac{d\phi}{ds} \right| = \frac{\sqrt{E \left(\frac{\partial\phi}{\partial v} \right)^2 - 2F \frac{\partial\phi}{\partial u} \frac{\partial\phi}{\partial v} + G \left(\frac{\partial\phi}{\partial u} \right)^2}}{\sqrt{EG - F^2}},$$

ans of representing graphically the magnitude of the differential quotient A for any direction is given by the following theorem

the tangent plane to a surface at a point M the positive half at M , corresponding to all values of k , positive and negative,

be drawn, and on them the corresponding lengths A be laid off from M , the locus of the extremities of these lengths is a circle tangent to the curve $\phi = \text{const}$

The proof of this theorem is simplified if we effect a transformation of curvilinear coordinates. Thus we take for the new coordinate lines the curves $\phi = \text{const}$ and their orthogonal trajectories. We let the former be denoted by $u_1 = \text{const}$ and the latter by $v_1 = \text{const}$, and indicate by subscript 1 functions in terms of these parameters. Now $F_1 = 0$, so that

$$A = \frac{du_1}{ds} = \frac{1}{\sqrt{E_1 + G_1 k_1^2}},$$

where k_1 denotes the value of dv_1/du_1 which determines a given direction, and the maximum length is $(E_1)^{-\frac{1}{2}}$. From (23) we have

$$\cos \theta_0 = \frac{\sqrt{E_1}}{\sqrt{E_1 + G_1 k_1^2}}, \quad \sin \theta_0 = \frac{\sqrt{G_1} k_1}{\sqrt{E_1 + G_1 k_1^2}},$$

where θ_0 is the angle which the given direction makes with the tangent to the curve $v_1 = \text{const}$. Hence if we regard the tangents at M to the curves $v_1 = \text{const}$ and $u_1 = \text{const}$ as axes of coordinates in the tangent plane, the coordinates of the end of a segment of length A are

$$\frac{\sqrt{E_1}}{E_1 + G_1 k_1^2}, \quad \frac{\sqrt{G_1} k_1}{E_1 + G_1 k_1^2}$$

The distance from this point to the mid-point of the maximum segment, measured along the tangent to $v_1 = \text{const}$, is readily found to be $\frac{1}{2\sqrt{E_1}}$, which proves the theorem.

37 Differential parameters of the first order If we put

$$(41) \quad \Delta_1 \phi = \frac{E \left(\frac{\partial \phi}{\partial v} \right)^2 - 2F \frac{\partial \phi}{\partial u} \frac{\partial \phi}{\partial v} + G \left(\frac{\partial \phi}{\partial u} \right)^2}{EG - F^2},$$

equation (40) can be written

$$\left(\frac{d\phi}{ds} \right)^2 = \Delta_1 \phi,$$

where now the differential quotient corresponds to the direction normal to the curve $\phi = \text{const}$. The left-hand member of this

is evidently independent of the nature of the parameters which the surface is referred to. Consequently the same holds for the right-hand member. Hence $\Delta_1\phi$ is unchanged in value when there is any change of parameters whatever. The significance of this result is as follows. Given a new set of parameters defined by $u=f_1(u_1, v_1)$, $v=f_2(u_1, v_1)$, let $\phi_1(u_1, v_1)$ be the result of substituting these expressions for u and v in ϕ and write the linear element thus

$$ds^2 = E_1 du_1^2 + 2 F_1 du_1 dv_1 + G_1 dv_1^2$$

The value of $\Delta_1\phi$ under this transformation is expressed by the following equation

$$\frac{2 F_1 \frac{\partial \phi}{\partial u} \frac{\partial \phi}{\partial v} + G_1 \left(\frac{\partial \phi}{\partial u} \right)^2}{E_1 G_1 - F_1^2} = \frac{E_1 \left(\frac{\partial \phi_1}{\partial v_1} \right)^2 - 2 F_1 \frac{\partial \phi_1}{\partial u_1} \frac{\partial \phi_1}{\partial v_1} + G_1 \left(\frac{\partial \phi_1}{\partial u_1} \right)^2}{E_1 G_1 - F_1^2}$$

It is left to the reader to verify this directly with the aid of (9). The invariant $\Delta_1\phi$ is called the *differential parameter of the first order*, this name and the notation are due to Lamé * or for the moment the partial differential equation

$$\Delta_1\phi = 0$$

is called the *line of curvature*. From the latter we get, by differentiation,

$$\frac{\partial \phi}{\partial u} du + \frac{\partial \phi}{\partial v} dv = 0$$

Replace $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ in (42) by dv and $-du$, which are respectively proportional to them, we obtain

$$E du^2 + 2 F du dv + G dv^2 = 0$$

The integral curves of equation (42) are lines of length zero, or, more precisely, if $\phi = \text{const}$ is a line of length zero, the function ϕ is constant on the integral curves of equation (42).

A particular case is that in which $\Delta_1\phi$ is a function of ϕ , say

$$\Delta_1\phi = F(\phi)$$

From (41) it is seen that when we put

$$\theta = \int \frac{d\phi}{\sqrt{F(\phi)}}$$

equation (43) becomes

$$(44) \quad \Delta_1 \theta = 1$$

As defined, θ is a function of ϕ , hence the family of curves $\theta = \text{const}$ is the same as the family $\phi = \text{const}$. Suppose we have such a family, and we take the curves $\theta = \text{const}$ for the curves $u = \text{const}$ and their orthogonal trajectories for $v = \text{const}$, thus effecting a change of parameters. Since $\Delta_1 u = 1$, it follows from (41) that $E = 1$, and consequently the linear element is

$$ds^2 = du^2 + G dv^2$$

Since now the linear element of a curve $v = \text{const}$ is du , the length of the curve between its points of intersection with two curves $u = u_0$ and $u = u_1$ is $u_1 - u_0$. Moreover, this length is the same for the segment of every curve $v = \text{const}$ between these two curves. For this reason the latter curves are said to be *parallel*. Conversely, in order that the curves $u = \text{const}$ of an orthogonal system be parallel, it is necessary that the linear element of the curves $v = \text{const}$ be independent of v . Hence E must be a function of u alone, which, by a transformation of coordinates, can be made equal to unity. Hence we have the theorem

A necessary and sufficient condition that the curves of a family $\phi = \text{const}$ be parallel is that $\Delta_1 \phi$ be a function of ϕ .

Let $\phi = \text{const}$ and $\psi = \text{const}$ be the equations of two curves upon a surface, through a point M , and let θ denote the angle between the tangents at M . If we put

$$(46) \quad \Delta_1(\phi, \psi) = \frac{E \frac{\partial \phi}{\partial v} \frac{\partial \psi}{\partial v} - F \left(\frac{\partial \phi}{\partial v} \frac{\partial \psi}{\partial u} + \frac{\partial \phi}{\partial u} \frac{\partial \psi}{\partial v} \right) + G \frac{\partial \phi}{\partial u} \frac{\partial \psi}{\partial u}}{EG - F^2},$$

the expression (21) for $\cos \theta$ can be written

$$(47) \quad \cos \theta = \frac{\Delta_1(\phi, \psi)}{\sqrt{\Delta_1 \phi} \sqrt{\Delta_1 \psi}}$$

is θ is an invariant for transformations of coordinates, it follows from this equation that $\Delta_1(\phi, \psi)$ also is an invariant. It is a *mixed differential parameter of the first order*. An immediate consequence of (47) is that

$$\Delta_1(\phi, \psi) = 0$$

in addition of orthogonality of the curves $\phi = \text{const.}$ and

equation (22) can be written

$$\sin \theta = \frac{1}{H\sqrt{\Delta_1\phi}\sqrt{\Delta_1\psi}} \left(\frac{\partial\phi}{\partial u} \frac{\partial\psi}{\partial v} - \frac{\partial\phi}{\partial v} \frac{\partial\psi}{\partial u} \right),$$

in terms of the function $\Theta(u, v)$, defined thus by Darboux,*

$$\Theta(\phi, \psi) = \frac{1}{H} \left(\frac{\partial\phi}{\partial u} \frac{\partial\psi}{\partial v} - \frac{\partial\phi}{\partial v} \frac{\partial\psi}{\partial u} \right),$$

written in the abbreviated form

$$\sin \theta = \frac{\Theta(\phi, \psi)}{\sqrt{\Delta_1\phi} \sqrt{\Delta_1\psi}}$$

If the functions in this identity except $\Theta(\phi, \psi)$ are known invariants, we have a proof that it also is an invariant. It is a *mixed differential parameter of the first order*. From (47) it follows that

$$\Delta_1^2(\phi, \psi) + \Theta^2(\phi, \psi) = \Delta_1\phi \Delta_1\psi,$$

hence the three invariants defined thus far are not independent of one another.

From (41) and (46) it follows that

$$\Delta_1 u = \frac{G}{H^2}, \quad \Delta_1(u, v) = \frac{-F}{H}, \quad \Delta_1 v = \frac{E}{H^2},$$

hence we find

$$\Theta(u, v) = \Delta_1 u \Delta_1 v - \Delta_1^2(u, v) = \frac{1}{H^2}$$

hence

$$E = \frac{\Delta_1 v}{\Theta^2(u, v)}, \quad F = \frac{-\Delta_1(u, v)}{\Theta^2(u, v)}, \quad G = \frac{\Delta_1 u}{\Theta^2(u, v)}$$

F , and G are differential invariants of the first order

Another result of these equations is the following. If the parameters of the surface are changed in accordance with the equations

$$u_1 = u_1(u, v), \quad v_1 = v_1(u, v),$$

and the resulting linear element is written,

$$ds^2 = E_1 du_1^2 + 2F_1 du_1 dv_1 + G_1 dv_1^2,$$

the value of E_1 is given by

$$E_1 = \frac{\Delta_1 v_1}{\Theta^2(u_1, v_1)} = \frac{E \left(\frac{\partial v_1}{\partial v} \right)^2 - 2F \frac{\partial v_1}{\partial u} \frac{\partial v_1}{\partial v} + G \left(\frac{\partial v_1}{\partial u} \right)^2}{\left[\frac{\partial(u_1, v_1)}{\partial(u, v)} \right]^2},$$

and F_1 and G_1 are found in like manner. In consequence of (51) these equations are equivalent to (13), which were found by direct calculation.

38 Differential parameters of the second order. Thus far we have considered differential invariants of the first order only. We introduce now one of the second order, discovered by Beltrami*. To this end we study the integral

$$\Pi = \iint \Delta_1(\phi, \psi) d\Sigma = \iint \Delta_1(\phi, \psi) H du dv,$$

for an ordinary portion of the surface bounded by a closed curve C (cf. § 33). For convenience we put

$$(53) \quad M = \frac{G \frac{\partial \psi}{\partial u} - F \frac{\partial \psi}{\partial v}}{H}, \quad N = \frac{E \frac{\partial \psi}{\partial v} - F \frac{\partial \psi}{\partial u}}{H},$$

so that, in consequence of (46), we have

$$\Pi = \iint \left(M \frac{\partial \phi}{\partial u} + N \frac{\partial \phi}{\partial v} \right) du dv$$

This may be written

$$\Pi = \iint \left[\frac{\partial}{\partial u} (M\phi) + \frac{\partial}{\partial v} (N\phi) \right] du dv - \iint \phi \left(\frac{\partial M}{\partial u} + \frac{\partial N}{\partial v} \right) du dv$$

If we apply Green's theorem to the first integral, this equation reduces to

$$(54) \quad \Pi = \int_C \phi (M dv - N du) - \iint \phi \left(\frac{\partial M}{\partial u} + \frac{\partial N}{\partial v} \right) du dv,$$

* *Ricerche di analisi applicata alla geometria, Giornale di matematiche*, Vol. II (1864), p. 365.

the first integral is curvilinear and is taken about C in the ordinary manner. Evidently du and dv refer to a displacement

If we indicate by δ variations in directions normal to C directed toward the interior of the contour, then from (23) it follows that

$$\frac{E du + F dv}{ds} = -\frac{H \delta v}{\delta s}, \quad \frac{F du + G dv}{ds} = \frac{H \delta u}{\delta s}$$

$$M dv - N du = \left(\frac{\partial \psi}{\partial u} \frac{\delta u}{\delta s} + \frac{\partial \psi}{\partial v} \frac{\delta v}{\delta s} \right) ds = \frac{\delta \psi}{\delta s} ds,$$

$$\Pi = \iint \Delta_1(\phi, \psi) d\Sigma = \int_c \phi \frac{\delta \psi}{\delta s} ds - \iint \phi \frac{1}{H} \left(\frac{\partial M}{\partial u} + \frac{\partial N}{\partial v} \right) d\Sigma$$

the terms in this equation, with the exception of $\frac{1}{H} \left(\frac{\partial M}{\partial u} + \frac{\partial N}{\partial v} \right)$, are independent of the choice of parameters. Hence the latter is invariant. It is called *the differential parameter of the second order* and is denoted by $\Delta_2 \psi$. In consequence of (53) we have

$$\Delta_2 \psi = \frac{1}{H} \left\{ \frac{\partial}{\partial u} \left(\frac{G \frac{\partial \psi}{\partial u} - F \frac{\partial \psi}{\partial v}}{H} \right) + \frac{\partial}{\partial v} \left(\frac{E \frac{\partial \psi}{\partial v} - F \frac{\partial \psi}{\partial u}}{H} \right) \right\}$$

In the foregoing discussion it has been assumed that only real surfaces appear. But all these results can be obtained directly by algebraic considerations of quadratic differential forms.*

Under any hypothesis regarding the character of the variables, the differential parameters can be used for any kind of curvilinear coordinates.

In addition to $\Delta_1 \phi$ there are other differential invariants of the second order, such as

$$\begin{aligned} \Delta_1 \Delta_1 \phi, & \quad \Delta_1(\phi, \Delta_1 \phi), & \quad \Theta(\phi, \Delta_1 \phi) \\ \Delta_1 \Delta_1(\phi, \psi), & \quad \Delta_1(\Delta_1 \phi, \Delta_1 \psi), & \quad \Theta(\Delta_1 \phi, \Delta_1 \psi) \end{aligned}$$

and invariants of the second order. In like manner we can group invariants of the third order, for instance,

$$\Delta_1 \Delta_1 \Delta_1 \phi, \quad \Delta_1 \Delta_1(\phi, \Delta_1 \phi), \quad \Delta_1 \Delta_2 \phi, \quad \Delta_2 \Delta_1 \phi,$$

These invariants and others, which can be obtained by an evident extension of this method, involve functions ϕ , ψ , E , F , G , and their derivatives

Conversely, we shall show * that every invariant of the form

$$I = f\left(E, F, G, \frac{\partial E}{\partial u}, \frac{\partial G}{\partial v}, \phi, \frac{\partial \phi}{\partial u}, \psi, \frac{\partial \psi}{\partial u}\right),$$

where ϕ , ψ , are independent functions, is expressible by means of the symbols Δ and Θ . Already we have seen that E , F , and G can be expressed in terms of $\Delta_1 u$, $\Delta_1 v$, and $\Delta_1(u, v)$. Moreover, from (48) it follows that

$$\frac{\partial \lambda}{\partial u} = H\Theta(\lambda, v), \quad \frac{\partial \lambda}{\partial v} = H\Theta(u, \lambda),$$

when λ is any function whatever. Hence all the terms in I can be expressed in terms of the symbols Δ and Θ , applied to

$$u, v, \phi, \psi,$$

Since u and v do not appear explicitly in I , we can effect a change of parameters, replacing u and v by ϕ and ψ respectively, and consequently we express I in terms of ϕ , ψ , and the differential invariants obtained by applying the operators Δ and Θ to these functions. In case ϕ is the only function appearing in I , we can take for ψ , in the change of parameters, any invariant of ϕ , such as $\Delta_1 \phi$ or $\Delta_2 \phi$, so long as it is not a function of ϕ , E , F , or G .

EXAMPLES

- 1 When the linear element of a surface is in the form

$$ds^2 = \lambda(du^2 + dv^2),$$

where λ is a function of u and v , both u and v are solutions of the equation $\Delta_2 \theta = 0$, the differential parameter being formed with respect to the right hand member

- 2 Show that on the surface

$$x = u \cos v, \quad y = u \sin v, \quad z = av + \phi(u),$$

the curves $u = \text{const}$ are parallel

- 3 When the linear element is in the form

$$ds^2 = \cos^2 \alpha du^2 + \sin^2 \alpha dv^2,$$

where α is a function of u and v , both u and v are solutions of the equation

$$\Delta_1(\theta, \Delta_1 \theta) = 2 \Delta_2 \theta (\Delta_1 \theta - 1)$$

* Cf Beltrami, *l c*, p. 357

the curves $\phi = \text{const}$, $\psi = \text{const}$. form an orthogonal system on a surface, then the displacement on the x -axis of any displacement on the surface is given by

$$dx = \frac{dx}{ds} \frac{d\psi}{\sqrt{\Delta\psi}} + \frac{dx}{d\sigma} \frac{d\phi}{\sqrt{\Delta\phi}},$$

and $d\sigma$ are the elements of length of the curves $\phi = \text{const}$, $\psi = \text{const}$.

and ϕ are any functions of u and v , then

$$\begin{aligned} \Delta_1 f &= \left(\frac{\partial f}{\partial u} \right)^2 \Delta_1 u + 2 \frac{\partial f}{\partial u} \frac{\partial f}{\partial v} \Delta_1(u, v) + \left(\frac{\partial f}{\partial v} \right)^2 \Delta_1 v, \\ \Delta_2(f, \phi) &= \frac{\partial f}{\partial u} \frac{\partial \phi}{\partial u} \Delta_1 u + \left(\frac{\partial f}{\partial u} \frac{\partial \phi}{\partial v} + \frac{\partial f}{\partial v} \frac{\partial \phi}{\partial u} \right) \Delta_1(u, v) + \frac{\partial f}{\partial v} \frac{\partial \phi}{\partial v} \Delta_1 v, \\ \Delta_2 f &= \frac{\partial f}{\partial u} \Delta_2 u + \frac{\partial f}{\partial v} \Delta_2 v + \frac{\partial^2 f}{\partial u^2} \Delta_1 u + 2 \frac{\partial^2 f}{\partial u \partial v} \Delta_1(u, v) + \frac{\partial^2 f}{\partial v^2} \Delta_1 v \end{aligned}$$

Symmetric coordinates We have seen that through every point on a surface there pass two minimal curves which lie entirely on the surface, and that these curves are defined by the differential

$$E du^2 + 2 F du dv + G dv^2 = 0$$

the equations of these curves be written

$$\alpha(u, v) = \text{const}, \quad \beta(u, v) = \text{const},$$

as from (42) that

$$\Delta_1(\alpha) = 0, \quad \Delta_1(\beta) = 0$$

in any parameters

$$E = \frac{\Delta_1 v}{\Theta^2(u, v)}, \quad F = -\frac{\Delta_1(u, v)}{\Theta(u, v)}, \quad G = \frac{\Delta_1 u}{\Theta(u, v)},$$

the curves $\alpha = \text{const}$, $\beta = \text{const}$, are taken as parametric, the corresponding coefficients E and G are zero, and consequently the linear element of the surface has the form

$$ds^2 = \lambda d\alpha d\beta,$$

In general, λ is a function of α and β . Conversely, as follows from (58), when the linear element has the form (59) equations (7) are satisfied and the parametric curves are minimal. The only transformations of coordinates which preserve this form of the linear element are those which leave the minimal lines invariant, that is

$$\begin{cases} \alpha_1 = F(\alpha), & \beta_1 = F_1(\beta), \\ \text{or} & \\ \alpha_1 = F(\beta), & \beta_1 = F_1(\alpha), \end{cases}$$

where F and F_1 are arbitrary functions. Whenever the linear element has the form (59), we say that the parameters are *symmetric*. The above results are given by the theorem

When α and β are symmetric coordinates of a surface, any two arbitrary functions of α and β respectively are symmetric coordinates, and they are the only ones

The general linear element of a surface can be written as the product of two factors, namely

$$(61) \quad ds^2 = \left(\sqrt{E} du + \frac{F + iH}{\sqrt{E}} dv \right) \left(\sqrt{E} du + \frac{F - iH}{\sqrt{E}} dv \right)$$

If t and t_1 denote integrating factors of the respective terms of the right-hand member of this equation, a pair of symmetric coordinates is given by the quadratures

$$(62) \quad \begin{cases} t \left(\sqrt{E} du + \frac{F + iH}{\sqrt{E}} dv \right) = d\alpha, \\ t_1 \left(\sqrt{E} du + \frac{F - iH}{\sqrt{E}} dv \right) = d\beta \end{cases}$$

When these values are substituted in (61), and the result is compared with (59), it is seen that $\lambda = \frac{1}{tt_1}$.

The first of equations (62) can be replaced by

$$t \sqrt{E} = \frac{\partial \alpha}{\partial v}, \quad t \frac{F + iH}{\sqrt{E}} = \frac{\partial \alpha}{\partial u}$$

Eliminating t from these equations, we have

$$(63) \quad \frac{E \frac{\partial \alpha}{\partial v} - F \frac{\partial \alpha}{\partial u}}{H} = i \frac{\partial \alpha}{\partial u}$$

If this equation be multiplied by $\frac{F - iH}{E}$, the result can be reduced to

$$(64) \quad \frac{F \frac{\partial \alpha}{\partial v} - G \frac{\partial \alpha}{\partial u}}{H} = i \frac{\partial \alpha}{\partial v}$$

These equations it follows that

$$6), \quad \frac{\partial}{\partial v} \left(\frac{E \frac{\partial \alpha}{\partial v} - F \frac{\partial \alpha}{\partial u}}{H} \right) = \frac{\partial}{\partial u} \left(\frac{F \frac{\partial \alpha}{\partial v} - G \frac{\partial \alpha}{\partial u}}{H} \right),$$

$$\Delta_2 \alpha = 0$$

readily found that β also satisfies this condition

isothermic and isometric parameters When the surface is l the coordinates also, the factors in (61) are conjugate y Hence the conjugate imaginary of t can be taken n this case α and β are conjugate imaginary also In l lows we assume that this choice has been made, and write

$$\alpha = \phi + i\psi, \quad \beta = \phi - i\psi$$

se values be substituted in (59), we get

$$ds^2 = \lambda (d\phi^2 + d\psi^2)$$

we see that the curves $\phi = \text{const}$ and $\psi = \text{const}$ form a rectangular system Moreover, the elements of arc of these are $\sqrt{\lambda} d\psi$ and $\sqrt{\lambda} d\phi$ respectively Consequently when the ds its $d\phi$ and $d\psi$ are taken equal, the four points (ϕ, ψ) , $(\phi + d\phi, \psi)$, $(\phi, \psi + d\psi)$, $(\phi + d\phi, \psi + d\psi)$ are the vertices of a square Hence the curves $\phi = \text{const}$ and $\psi = \text{const}$ divide the surface into a network of small squares On this account the curves are called *isometric curves*, and ϕ and ψ *isometric parameters* These lines are of importance in the theory of heat, termed *isothermal* or *isothermic*, which names are used in connection as synonymous with *isometric*

Further the linear element can be put in the symmetric form, and similar to (66) give at once a set of isometric parameters Conversely, the knowledge of a set of isometric parameters leads to a set of symmetric parameters But we have seen that when one set of symmetric parameters is known, all the others are determined by equations of the form (60) Hence we have the theorem

any pair of real isometric parameters ϕ, ψ for a surface, is accompanied by a pair ϕ_1, ψ_1 is given by equations of the form

$$\phi_1 + i\psi_1 = F(\phi \pm i\psi), \quad \phi_1 - i\psi_1 = F_0(\phi \mp i\psi),$$

and F_0 are any functions conjugate imaginary to one another

Consider, for instance, the case

$$(68) \quad \phi_1 + \imath \psi_1 = F(\phi + \imath \psi)$$

From the Cauchy-Riemann differential equations

$$(69) \quad \frac{\partial \phi_1}{\partial \phi} = \frac{\partial \psi_1}{\partial \psi}, \quad \frac{\partial \phi_1}{\partial \psi} = -\frac{\partial \psi_1}{\partial \phi},$$

it follows that ϕ_1 and ψ_1 are functions of both ϕ and ψ . Hence the curves $\phi_1 = \text{const}$, $\psi_1 = \text{const}$ are different from the system $\phi = \text{const}$, $\psi = \text{const}$. Similar results hold when $+\imath$ is replaced by $-\imath$ in the argument of the right-hand member of (68). Hence

There is a double infinity of isometric systems of lines upon a surface, when one system is known all the others can be found directly

If the value (66) for α be substituted in the first of equations (57), the resulting equation is reducible to

$$\Delta_1 \phi - \Delta_1 \psi + 2\imath \Delta_1(\phi, \psi) = 0$$

Since ϕ and ψ are real, this equation is equivalent to

$$(70) \quad \Delta_1 \phi = \Delta_1 \psi, \quad \Delta_1(\phi, \psi) = 0$$

From (58) it is seen that these equations are the condition that $E = G$, $F = 0$, when ϕ and ψ are the parameters. Hence equations (70) are the necessary and sufficient conditions that ϕ and ψ be isometric parameters.

Again, when α in (65) is replaced by $\phi + \imath \psi$, and all the functions are real, we have

$$(71) \quad \Delta \phi = 0, \quad \Delta \psi = 0$$

Conversely, when we have a function ϕ satisfying the first of these equations, the expression

$$\frac{F \frac{\partial \phi}{\partial u} - E \frac{\partial \phi}{\partial v}}{H} du + \frac{G \frac{\partial \phi}{\partial u} - F \frac{\partial \phi}{\partial v}}{H} dv$$

is an exact differential. Call it $d\psi$, then

$$(72) \quad \frac{F \frac{\partial \phi}{\partial u} - E \frac{\partial \phi}{\partial v}}{H} = \frac{\partial \psi}{\partial u}, \quad \frac{G \frac{\partial \phi}{\partial u} - F \frac{\partial \phi}{\partial v}}{H} = \frac{\partial \psi}{\partial v}$$

the equations be solved for $\frac{\partial \phi}{\partial u}$, $\frac{\partial \phi}{\partial v}$, we get

$$\frac{E \frac{\partial \psi}{\partial v} - F \frac{\partial \psi}{\partial u}}{H} = \frac{\partial \phi}{\partial u}, \quad \frac{F \frac{\partial \psi}{\partial v} - G \frac{\partial \psi}{\partial u}}{H} = \frac{\partial \phi}{\partial v}$$

to express the condition $\frac{\partial}{\partial v} \left(\frac{\partial \phi}{\partial u} \right) = \frac{\partial}{\partial u} \left(\frac{\partial \phi}{\partial v} \right)$, we find that

Moreover, these two functions ϕ and ψ satisfy (70), consequence of (72) and (73), and therefore they are isometric curves. Hence

A necessary and sufficient condition that ϕ be the isometric parameter family of an isometric system on a surface is that $\Delta_s \phi = 0$, the isometric parameter of the other family is given by a quadrature

Actually we remark that if u and v are a pair of isometric curves, equations (72) and (73) reduce to (69)

isothermic orthogonal systems If the linear element of a surface is given in the form (67) and the parameters are changed in accordance with the equations

$$\phi = f_1(u), \quad \psi = f_2(v),$$

the linear element becomes

$$ds^2 = \lambda (f_1'^2 du + f_2'^2 dv^2),$$

where accents indicate differentiation. However, this transformation of parameters has not changed the coordinate lines, the coefficients are now no longer equal, but in the relation

$$\frac{U}{G} = \frac{V}{F},$$

and U and V denote functions of u and v respectively. When this relation is satisfied the linear element can be written

$$ds^2 = \lambda (U du^2 + V dv^2),$$

the transformation of coordinates,

$$\phi = \int \sqrt{U} du, \quad \psi = \int \sqrt{V} dv,$$

it is brought to the form (67), whatever be U and V , and the coordinate lines are unaltered. Hence

A necessary and sufficient condition that an orthogonal system of parametric lines on a surface form an isothermic system is that the coefficients of the corresponding linear element satisfy a relation of the form (74)

We seek now the necessary and sufficient condition which a function $\omega(u, v)$ must satisfy in order that the curves $\omega = \text{const}$ and their orthogonal trajectories form an isothermic system. Either ω , or a function of it, is the isothermic parameter of the curves $\omega = \text{const}$. We denote this parameter by ϕ , then $\phi = f(\omega)$. Since ϕ must be a solution of equations (71), we have, on substitution,

$$\Delta_2 \omega f'(\omega) + \Delta_1 \omega f''(\omega) = 0,$$

where the primes indicate differentiation with respect to ω . If this equation is written in the form

$$\frac{\Delta_2 \omega}{\Delta_1 \omega} = - \frac{f''(\omega)}{f'(\omega)},$$

we see that the ratio of the two differential parameters is a function of ω . Conversely, if this ratio is a function of ω , the function $f(\omega)$, obtained by two quadratures from

$$(77) \quad f'(\omega) = e^{-\int \frac{\Delta_2 \omega}{\Delta_1 \omega} d\omega},$$

will satisfy equations (71). Hence

A necessary and sufficient condition that a family of curves $\omega = \text{const}$ and their orthogonal trajectories form an isothermic system is that the ratio of $\Delta_2 \omega$ and $\Delta_1 \omega$ be a function of ω

Suppose we have such a function ω , then the orthogonal trajectories of the curves $\omega = \text{const}$ can be found by quadrature, for, the differential equation of these trajectories is

$$(78) \quad \left(E \frac{\partial \omega}{\partial v} - F \frac{\partial \omega}{\partial u} \right) du + \left(F \frac{\partial \omega}{\partial v} - G \frac{\partial \omega}{\partial u} \right) dv = 0$$

If equation (76) be written in the form

$$\frac{\partial}{\partial v} \left[f'(\omega) \frac{E \frac{\partial \omega}{\partial v} - F \frac{\partial \omega}{\partial u}}{H} \right] + \frac{\partial}{\partial u} \left[f'(\omega) \frac{G \frac{\partial \omega}{\partial v} - F \frac{\partial \omega}{\partial u}}{H} \right] = 0,$$

n that an integrating factor of equation (78) is $f'(\omega)/H$, $f'(\omega)$ is given by (77). Hence $f(\omega)$ and the function ψ , by the quadrature

$$\frac{\partial \psi}{\partial u} = -f'(\omega) \frac{E \frac{\partial \omega}{\partial v} - F \frac{\partial \omega}{\partial u}}{H}, \quad \frac{\partial \psi}{\partial v} = f'(\omega) \frac{G \frac{\partial \omega}{\partial u} - F \frac{\partial \omega}{\partial v}}{H},$$

ir of isometric parameters. From these equations and (77) s that

$$\Delta_1 \psi = e^{-2 \int \frac{\Delta_{2\omega}}{\Delta_1 \omega} d\omega} \Delta_1 \omega,$$

sequently, by means of (52), the linear element can be ie form

$$ds^2 = \frac{1}{\Delta_1 \omega} \left(d\omega^2 + e^{2 \int \frac{\Delta_{2\omega}}{\Delta_1 \omega} d\omega} d\psi^2 \right)$$

ear element of the plane referred to rectangular axes is $ds^2 = dx^2 + dy^2$ itly x and y are isothermic parameters, and we have the theorem

the curves whose equations are obtained by equating to constants the real nary parts of any function of $x + iy$ or $x - iy$ form an isothermal orthogon, and every such system can be obtained in this way

ample, consider $\phi + i\psi = \frac{c^2}{x - iy},$

any constant. From this it follows that

$$\phi = \frac{c^2 x}{x^2 + y^2}, \quad \psi = \frac{c^2 y}{x^2 + y^2}$$

circles $\phi = \text{const}$, $\psi = \text{const}$ form an isothermal orthogonal system, ψ are isothermic parameters.

ove system of circles is a particular case of the system considered in § 34 e whether the latter also form an isothermal system. If we put

$$\omega = x + \frac{1}{x}(y^2 - a^2),$$

at $\Delta_1 \omega = \frac{1}{x^2}(\omega^2 + 4a^2), \quad \Delta_2 \omega = \frac{2\omega}{x^2}$

ratio of $\Delta_1 \omega$ and $\Delta_2 \omega$ is a function of ω , and consequently the system of isothermal. From (77) it follows that the isothermic parameter of the y is $\phi = \frac{1}{2a} \tan^{-1} \frac{\omega}{2a}$, and the parameter of the orthogonal family is

$$\psi = \frac{1}{2a} \tanh^{-1} \frac{\bar{\omega}}{2a}, \quad \bar{\omega} = y + \frac{x^2 + a^2}{y}$$

EXAMPLES

1 Show that the meridians and parallels on a sphere form an isothermal orthogonal system, and determine the isothermic parameters

2 Show that a system of confocal ellipses and hyperbolas form an isothermal orthogonal system in the plane

3 Show that the surface

$$\frac{x}{a} = \sqrt{\frac{(a^2 - u)(a^2 - v)}{(a^2 - b^2)(a^2 - c^2)}}, \quad \frac{y}{b} = \sqrt{\frac{(b^2 - u)(b^2 - v)}{(b^2 - a^2)(b^2 - c^2)}}, \quad \frac{z}{c} = \sqrt{\frac{(c^2 - u)(c^2 - v)}{(c^2 - a^2)(c^2 - b^2)}}$$

is an ellipsoid, and that the parametric curves form an isothermal orthogonal system

4 Find the curves which bisect the angles between the parametric curves on the surface

$$\frac{x}{a} = \frac{u+v}{2}, \quad \frac{y}{b} = \frac{u-v}{2}, \quad z = \frac{uv}{2},$$

show that they form an isothermal orthogonal system

5 Determine $\phi(v)$ so that on the right conoid $x = u \cos v$, $y = u \sin v$, $z = \phi(v)$ the parametric curves form an isothermal orthogonal system, and show that the curves which bisect the angles between the parametric curves form a system of the same kind

6 Express the results of Ex 4, page 82, in terms of the parameters ϕ and ψ defined by (66)

42 Conformal representation When a one-to-one correspondence of any kind is established between the points of two surfaces, either surface may be said to be *represented* on the other. Thus, if we roll out a cylindrical surface upon a plane and say that the points of the surface correspond to the respective points of the plane into which they are developed, we have a representation of the surface upon the plane. Furthermore, as there is no stretching or folding of the surface in this development of it upon the plane, lengths of lines and the magnitude of angles are unaltered. It is evidently impossible to make such a representation of every surface upon a plane, and, in general, two surfaces of this kind do not admit of such a representation upon one another. However, it is possible, as we shall see, to represent one surface upon another in such a way that the angles between corresponding lines on the surfaces are equal. In this case we say that one surface has *conformal representation* on the other.

In order to obtain the condition to be satisfied for a conformal representation of two surfaces S and S' , we imagine that they are referred to a corresponding system of real lines in terms of the

parameters u, v , and that corresponding points have the same linear elements. We write their linear elements in the following forms

$$E du^2 + 2F du dv + G dv^2, \quad ds^2 = E' du'^2 + 2F' du' dv' + G' dv'^2$$

The angles ω and ω' between the coordinate lines at corresponding points must be equal, it is necessary that

$$\frac{F}{\sqrt{EG}} = \frac{F'}{\sqrt{E'G'}}$$

Let θ_0 denote the angles which a curve on S and the corresponding curve on S' respectively make with the curves $v = \text{const.}$ of the former curves, we have, from (23) and (25),

$$\begin{aligned} \sin \theta_0 &= \frac{H}{\sqrt{E}} \frac{dv}{ds}, & \sin(\omega - \theta_0) &= \frac{H}{\sqrt{G}} \frac{du}{ds}, \\ \sin \theta'_0 &= \frac{H'}{\sqrt{E'}} \frac{dv'}{ds'}, & \sin(\omega' - \theta'_0) &= \frac{H'}{\sqrt{G'}} \frac{du'}{ds'} \end{aligned}$$

whence $\omega' = \pm \omega$ and $\theta'_0 = \pm \theta_0$, according as the angles have the same or opposite sense. Hence we have

$$\frac{H'}{\sqrt{E'}} \frac{dv'}{ds'} = \pm \frac{H}{\sqrt{E}} \frac{dv}{ds}, \quad \frac{H'}{\sqrt{G'}} \frac{du'}{ds'} = \pm \frac{H}{\sqrt{G}} \frac{du}{ds},$$

regarding the sense of the angles. From these equations we find

$$\frac{\sqrt{E}}{\sqrt{G}} = \frac{\sqrt{E'}}{\sqrt{G'}},$$

in combination with (81), may be written

$$\frac{L'}{L} = \frac{F'}{F} = \frac{G'}{G} = t^2,$$

where t^2 denotes the factor of proportionality, a function of u and v alone. From (83) it follows at once that

$$ds'^2 = t^2 ds^2$$

when the proportion (83) is satisfied, the equations (81) follow. Hence we have the theorem

Necessary and sufficient condition that the representation of two surfaces referred to a corresponding system of lines be conformal is

that the first fundamental coefficients of the two surfaces be proportional, the factor of proportionality being a function of the parameters, the representation is direct or inverse according as the relative positions of the positive half tangents to the parametric curves on the two surfaces are the same or symmetric

Later we shall find means of obtaining conformal representations

From (84) it follows that small arcs measured from corresponding points on S and S' along corresponding curves are in the same ratio, the factor of proportionality being in general a function of the position of the point. Conversely, when the ratio is the same or all curves at a point, there is a relation such as (84), with t a function of u and v at most. And since it holds for all directions, we must have the proportion (83). On this account we may say that two surfaces are represented conformally upon one another when in the neighborhood of each pair of homologous points corresponding small lengths are proportional.

43 Isometric representation. Applicable surfaces. When in particular the factor t is equal to unity, corresponding small lengths are equal as well as angles. In this case the representation is said to be *isometric*, and the two surfaces are said to be *applicable*. The significance of the latter term is that the portion of one surface in the neighborhood of every point can be so bent as to be made to coincide with the corresponding portion of the other surface without stretching or duplication. It is evident that such an application of one surface upon another necessitates a continuous array of surfaces applicable to both S and S_1 . This process of transformation is called *deformation*, and S_1 is called a *deform* of S and vice versa. An example of this is afforded by the rolling of a cylinder on a plane.

Although a conformal representation can be established between any two surfaces, it is not true, as we shall see later, that any two surfaces admit of an isometric representation upon one another. From time to time we shall meet with examples of applicable surfaces, and in a later chapter we shall discuss at length problems which arise concerning the applicability of surfaces. However, we consider here an example afforded by the tangent surface of a twisted curve.

all that if x, y, z are the coordinates of a point on the surface expressed in terms of the arc, the equations of the surface are

$$\xi = x + x't, \quad \eta = y + y't, \quad \zeta = z + z't,$$

near element of the surface is

$$d\sigma^2 = \left(1 + \frac{t^2}{\rho^2}\right) ds^2 + 2 ds dt + dt^2,$$

denotes the radius of curvature of the curve.

This expression does not involve the radius of torsion, it has that the tangent surfaces to all curves which have the intrinsic equation $\rho = f(s)$ are applicable in such a way that the curves determined by the same value of s correspond. If C is a plane curve with this equation, the surface is applicable to the plane in such a way that points of the surface corresponding to a curve are the characteristics of the osculating plane at the point of osculation moves along the curve, and consequently they are the axes of rotation of the osculating plane as it envelopes the surface. Instead of rolling the plane over the surface, we may roll the surface over the plane and bring points into coincidence with the plane. It is in this sense that the surface is developable upon a plane, and for this reason we call a developable surface (cf § 27). Later it will be shown that every surface applicable to the plane is the tangent surface of a curve (§ 64).

Conformal representation of a surface upon itself. We return to the consideration of conformal representation, and remark that a consequence of equations (83) is that the minimal curves correspond upon S and S' . Conversely, when two surfaces are related by a corresponding system of lines, if the minimal lines on the two surfaces correspond, equations (83) must hold. Hence we have the necessary and sufficient condition that the representation of two surfaces upon one another be conformal is that the minimal lines correspond.

If the minimal lines upon the two surfaces are known and taken as principal lines, the linear elements are of the form

$$ds^2 = \lambda da d\beta, \quad ds'^2 = \lambda_1 da_1 d\beta_1$$

Hence a conformal representation is defined in the most general way by the equations

$$(86) \quad \alpha_1 = F(\alpha), \quad \beta_1 = F_1(\beta),$$

or

$$(87) \quad \alpha_1 = F(\beta), \quad \beta_1 = F_1(\alpha),$$

where F and F_1 are arbitrary functions which must be conjugate imaginary when the surfaces are real

Instead of interpreting (85) as the linear elements of two surfaces referred to their minimal lines, we can look upon them as the linear element of the same surface in terms of two sets of parameters referring to the minimal lines. From this point of view equations (86) and (87) define the most general conformal representation of a surface upon itself. If we limit our consideration to real surfaces and put, as before,

$$\alpha = \phi + i\psi, \quad \beta = \phi - i\psi, \quad \alpha_1 = \phi_1 + i\psi_1, \quad \beta_1 = \phi_1 - i\psi_1,$$

the functions ϕ , ψ and ϕ_1 , ψ_1 are pairs of isothermic parameters. Now equations (86), (87) may be written

$$(88) \quad \phi_1 + i\psi_1 = F(\phi \pm i\psi)$$

Consequently we have the theorem

When a pair of isothermic parameters ϕ , ψ of a surface are known and the surface is referred to the lines $\phi = \text{const}$, $\psi = \text{const}$, the most general conformal representation of the surface upon itself is obtained by making a point (ϕ, ψ) correspond to the point (ϕ_1, ψ_1) into which it can be transformed in accordance with equation (88)

As a corollary of this theorem, we have

When a pair of isothermic parameters is known for each of two surfaces, all the conformal representations of one surface upon the other can be found directly

Consider two pairs of isothermic parameters ϕ , ψ and ϕ_1 , ψ_1 for a surface S , and suppose their relation is

$$(89) \quad \phi_1 + i\psi_1 = F(\phi + i\psi)$$

If two curves C and C_1 are in correspondence in this representation, their parametric equations must be the same functional relation between the parameters, namely,

$$f(\phi, \psi) = 0, \quad f(\phi_1, \psi_1) = 0$$

by θ and θ_1 the angles which C and C_1 make with the $v = \text{const}$ and $\psi_1 = \text{const}$ respectively. If we write the element of S in the two forms

$$ds^2 = \lambda^2 (d\phi^2 + d\psi^2), \quad ds_1^2 = \lambda_1^2 (d\phi_1^2 + d\psi_1^2),$$

from (23) that

$$\cos \theta = \frac{d\phi}{\sqrt{d\phi^2 + d\psi^2}}, \quad \sin \theta = \frac{d\psi}{\sqrt{d\phi^2 + d\psi^2}},$$

$$\cos \theta_1 = \frac{d\phi_1}{\sqrt{d\phi_1^2 + d\psi_1^2}}, \quad \sin \theta_1 = \frac{d\psi_1}{\sqrt{d\phi_1^2 + d\psi_1^2}}$$

these expressions we derive the following

$$e^{2i\theta} = \frac{d\phi + i d\psi}{d\phi - i d\psi}, \quad e^{2i\theta_1} = \frac{d\phi_1 + i d\psi_1}{d\phi_1 - i d\psi_1},$$

in consequence of (89) we have

$$e^{2i(\theta_1 - \theta)} = \frac{F'(\phi + i\psi)}{F'_0(\phi - i\psi)},$$

F_0 is the function conjugate to F , and the accents indicate differentiation with respect to the argument. If Γ and Γ_1 are a pair of corresponding curves, and their angles are denoted by $\bar{\theta}_1$, it follows from (90) that

$$e^{2i(\bar{\theta}_1 - \bar{\theta})} = e^{-2i(\theta_1 - \theta)},$$

$$\bar{\theta}_1 - \theta_1 = \bar{\theta} - \theta$$

the right-hand member of (90) is merely a function of the point and is independent of directions. Hence in conformal representation defined by an equation of the form (89) the angles between corresponding curves have the same sense, and, now, the correspondence satisfies the equation

$$\phi_1 + i\psi_1 = F(\phi - i\psi),$$

an equation analogous to (90) is

$$e^{2i(\theta_1 + \theta)} = \frac{F'(\phi - i\psi)}{F'_0(\phi + i\psi)}$$

$$\theta_1 - \bar{\theta}_1 = \bar{\theta} - \theta,$$

hence the corresponding angles are equal in the inverse sense

45 Conformal representation of the plane For the plane the preceding theorem may be stated thus

The most general real conformal representation of the plane upon itself is obtained by making a point (x, y) correspond to the point (x_1, y_1) , where $x_1 + iy_1$ is any function of $x + iy$ or $x - iy$

We recall the example of § 41, namely

$$(i) \quad x_1 + iy_1 = \frac{c^2}{x - iy},$$

where c is a real constant. This equation is equivalent to

$$(ii) \quad x = \frac{c^2 x_1}{x_1^2 + y_1^2}, \quad y = \frac{c^2 y_1}{x_1^2 + y_1^2}, \quad (x^2 + y^2)(x_1^2 + y_1^2) = c^4,$$

and also to

$$(iii) \quad x_1 = \frac{c^2 x}{x^2 + y^2}, \quad y_1 = \frac{c^2 y}{x^2 + y^2}$$

Hence the parallels $x = \text{const}$ and $y = \text{const}$, in the xy -plane, are represented in the x_1y_1 -plane by circles which pass through the origin and have their centers on the respective axes. Conversely, these circles in the xy plane correspond to the parallels in the x_1y_1 -plane.

If we put

$$x^2 + y^2 = r^2, \quad x_1^2 + y_1^2 = r_1^2,$$

equations (ii) and (iii) may be written

$$(iv) \quad \frac{x}{r} = \frac{x_1}{r_1}, \quad \frac{y}{r} = \frac{y_1}{r_1}, \quad rr_1 = c^2$$

Hence corresponding points are on the same line through the origin, and their distances from it are such that $rr_1 = c^2$. On this account equations (iv) are said to define an *inversion with respect to the circle $x^2 + y^2 = c^2$* , or, since $r_1 = c^2/r$, *a transformation by reciprocal radii vectors*.

From § 44 it follows that corresponding angles are equal in the inverse sense.

For the case

$$(v) \quad x_1 + iy_1 = \frac{c^2}{x + iy}$$

the equations analogous to (iv) are

$$\frac{x}{r} = \frac{x_1}{r_1}, \quad \frac{y}{r} = -\frac{y_1}{r_1}$$

Hence the point $P_1(x_1, y_1)$ corresponding to $P(x, y)$ lies on the line which is the reflection in the x -axis of the line OP , and at the distance $OP_1 = c^2/r$. Evidently this transformation is the combination of an inversion and the transformation $x_1 = x, y_1 = -y$.

One finds that the transformations (i) and (v) have the following properties

Every straight line is transformed into a circle which passes through the origin, and conversely

Every circle which does not pass through the origin is transformed into a circle

propose now the problem of finding the most general transformation of the plane into itself, which changes *not passing through the origin* into circles. In solving it the plane to symmetric parameters α, β , where

$$\alpha = x + iy, \quad \beta = x - iy$$

ation of any circle which does not pass through the origin of the form

$$c\alpha\beta + a\alpha + b\beta + d = 0,$$

, b, c, d are constants, when the circle is real a and b conjugate imaginaries and c real. Equation (91) defines β as a function of α . If we differentiate the equation three times with respect to α , and eliminate the constants from the resulting equations, we find

$$3\beta''^2 - 2\beta'\beta''' = 0,$$

where the accent indicates differentiation with respect to α . Since, as equation (91) contains three independent constants, the general integral of (92)

shows that the most general conformal representation of the plane upon itself is given by

$$\alpha_1 = A(\alpha), \quad \beta_1 = B(\beta),$$

$$\alpha_1 = B(\beta), \quad \beta_1 = A(\alpha)$$

the problem reduces, therefore, to the determination of functions A, B , such that the equation

$$3\beta_1''^2 - 2\beta_1'\beta_1''' = 0,$$

where the accent indicates differentiation with respect to α_1 , can be transformed by (93) or (94) into (92)

Consider first equations (93), which we write

$$\alpha = A_1(\alpha_1), \quad \beta = B(\beta)$$

$$\beta_1' = \frac{d\beta}{d\alpha} \frac{d\beta}{d\alpha} \frac{d\alpha}{d\alpha_1} = B'\beta' A_1'$$

In this manner we find β_1'' and β_1''' . When their values are substituted in (95) we get, since A_1' and B' are different from zero,

$$2\beta'\beta''' + \frac{1}{B'^2} (3B''^2 - 2B'B''')\beta'^4 + \frac{1}{A_1'^4} (3A_1''^2 - 2A_1'A_1''')\beta'^2 = 0$$

Since equation (95) must be directly transformable into (92), it follows that

$$(96) \quad 3B''^2 - 2B'B''' = 0, \quad 3A_1''^2 - 2A_1'A_1''' = 0$$

As these equations are of the form (92), their general integrals are similar to (91). Hence the most general forms of (93) for our problem are

$$(97) \quad \alpha_1 = \frac{a_1\alpha + a_2}{a_3\alpha + a_4}, \quad \beta_1 = \frac{b_1\beta + b_2}{b_3\beta + b_4}$$

Moreover, when these values are substituted in an equation in α_1, β_1 of the form (91), the resulting equation in α and β is of this form

Equation (91) may likewise be looked upon as defining α in terms of β , so that α , as a function of β , satisfies an equation of the form (92), similarly for α_1 as a function of β_1 . Hence if we had used (94), we should have been brought to results analogous to (97), and therefore the most general forms of (94) for our problem are

$$(98) \quad \alpha_1 = \frac{b_1\beta + b_2}{b_3\beta + b_4}, \quad \beta_1 = \frac{a_1\alpha + a_2}{a_3\alpha + a_4}$$

Hence

*When a plane is defined in symmetric parameters α, β , the most general conformal representation of the plane upon itself, for which circles correspond to circles or straight lines, is given by (97) or (98) **

EXAMPLES

1 Deduce the equations which define the most general conformal representation of a surface with the linear element $ds^2 = du^2 + (a^2 - u)dv^2$ upon itself

2 Show that the surfaces

$$\begin{aligned} x &= u \cos v, & y &= u \sin v, & z &= av, \\ x &= u \cos v, & y &= u \sin v, & z &= a \cosh \frac{u}{a} \end{aligned}$$

are applicable. Find the curve in which a plane through the z -axis cuts the latter surface, and deduce the equations of the conformal representation of these surfaces on the plane

3 When the representation is defined by (97), what are the coordinates of the center and radius of the circle in the α_1 plane which corresponds to the circle of center (c, d) and radius r in the α plane?

* The transformations (97) and (98) play an important role in the theory of functions. For a more detailed study of them the reader is referred to the treatises of Picard, Darboux, and Forsyth.

γ that in the conformal representation (97) there are, in general, two points, each of which corresponds to itself, also that if γ and δ are the α at these points, then

$$\frac{\gamma_1 - \gamma}{\gamma_1 - \delta} = \frac{\alpha - \gamma}{\alpha - \delta} K, \quad \text{where } K = \frac{a_1 + a_4 - \sqrt{(a_1 - a_4)^2 + 4a_2a_3}}{a_1 + a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}}$$

the condition that the origin be the only point which corresponds to itself, that if the quantities a_1, a_2, a_3, a_4 are real, a circle in the α -plane through O corresponds to a circle in the α_1 -plane through O and touching the other so that a circle touching the x -axis at O corresponds to itself

equation $2\alpha_1 = (a - b)\alpha + \frac{a + b}{\alpha}$, where a and b are constants, defines a representation of the plane upon itself, such that circles about the origin and lines through the latter in the α -plane correspond to confocal ellipses and hyperbolas in the α_1 plane

In the conformal representation $\alpha_1 = \log \alpha$ to lines parallel to the x - and y -axes in the α_1 -plane there correspond lines through the origin and circles about the origin in the α -plane, and to any orthogonal system of straight lines in the α_1 -plane there correspond a system of logarithmic spirals in the α plane

Surfaces of revolution By definition a *surface of revolution* is a surface generated by a plane curve when the plane of the curve is made to rotate about a line in the plane. The various parts of the curve are called the *meridians* of the surface, and the lines described by each point of the curve in the revolution are called the *parallels*. We take the axis of rotation for the z -axis, the x -axis and y axis any two lines perpendicular to one another, the z -axis, and meeting it in the same point. For any position of the plane the equation of the curve may be written $z = \phi(r)$, where r denotes the distance of a point of the curve from the z axis, and ϕ denotes the angle which the plane, in any of its positions, makes with the zx -plane. Hence the equations of the surface are

$$x = r \cos v, \quad y = r \sin v, \quad z = \phi(r)$$

A surface element is

$$ds' = [1 + \phi''(r)] dr + r dv$$

or, if

$$u = \int \frac{1}{r} \sqrt{1 + \phi'^2} dr,$$

the surface element is transformed into

$$ds' = \lambda (du^2 + dv^2),$$

where λ is a function of u , which shows that the meridians and parallels form an isothermal system. As this change of parameters does not change the parametric lines, the equations

$$x = u, \quad y = v,$$

define a conformal representation of the surface of revolution upon the plane in which the meridians and parallels correspond to the straight lines $x = \text{const}$ and $y = \text{const}$ respectively.

By definition a *loxodromic curve* on a surface of revolution is a curve which cuts the meridians under constant angle. Evidently it is represented on the plane by a straight line. Hence loxodromic curves on a surface of revolution (99) are given by

$$a \int \frac{1}{r} \sqrt{1 + \phi'^2} dr + bv + c = 0,$$

where a, b, c are constants.

Incidentally we have the theorem

When the linear element of a surface is reducible to the form

$$ds^2 = \lambda (du^2 + dv^2),$$

where λ is a function of u or v alone, the surface is applicable to a surface of revolution.

For, suppose that λ is a function of u alone. Put $r = \sqrt{\lambda}$ and solve this equation for u as a function of r . If the resulting expression be substituted in (101), we find, by a quadrature, the function $\phi(r)$, for which equations (99) define the surface of revolution with the given linear element.

When, in particular, the surface of revolution is the unit sphere, with center at the origin, we have

$$r = \sin u, \quad z = \sqrt{1 - r^2} = \cos u,$$

where u is the angle which the radius vector of the point makes with the positive z axis. Now

$$u_1 = \int \frac{1}{r} \sqrt{1 + \phi'^2} dr = \log \tan \frac{u}{2}$$

Hence the equations of correspondence are

$$x = \log \tan \frac{u}{2}, \quad y = v$$

representation is called a *Mercator chart* of the sphere upon the plane. It is used in making maps of the earth for mariners. A straight line on the chart cuts the meridians at right angle.

Conformal representations of the sphere We have found that when the unit sphere, with center at the origin, is transformed into minimal lines, its equations are

$$x = \frac{\alpha + \beta}{\alpha\beta + 1}, \quad y = i \frac{(\beta - \alpha)}{\alpha\beta + 1}, \quad z = \frac{\alpha\beta - 1}{\alpha\beta + 1},$$

and β are conjugate imaginary. Hence the parametric equations of any real circle on the sphere is of the form

$$c\alpha\beta + a\alpha + b\beta + d = 0,$$

and b are conjugate imaginary and c and d are real. It follows that the problem of finding any conformal transformation of the sphere upon the plane with circles of the sphere in correspondence with circles or straight lines of the plane is the same problem analytically as the determination of the conformal representation of the plane upon itself. Hence, from the results of § 45, it follows that

Conformal representations of the sphere (103) upon a plane, the lower signs of the former corresponding to circles or straight lines of the plane, are defined by

$$x_1 \pm iy_1 = \frac{a_1\alpha + a_2}{a_3\alpha + a_4}, \quad x_1 \mp iy_1 = \frac{b_1\beta + b_2}{b_3\beta + b_4} \quad *$$

We wish to consider in particular the case in which the sphere is represented on the xy plane in such a way that the great circle determined by this plane corresponds with itself point for point.

(103) we have that the equations of this circle are

$$\alpha\beta = 1, \quad x = \frac{\alpha^2 + 1}{2\alpha}, \quad y = i \frac{1 - \alpha^2}{2\alpha}$$

The representation with the lower signs is the combination of the one with the upper signs and the transformation $\alpha_1 = \beta$, $\beta_1 = \alpha$, which from (103) is seen to transform a figure into the figure symmetrical with respect to the yz plane.

When these values are substituted in (104) it is found that we must have $a_1 = a_4$, $b_1 = b_4$, $a_2 = a_3 = b_2 = b_3 = 0$,

so that the particular form of (104)* is equivalent to

$$x_1 = \frac{1}{2}(\alpha + \beta), \quad y_1 = \frac{1}{2}(\beta - \alpha)$$

From these equations and (103) we find that the equations of the straight lines joining corresponding points on the sphere and plane are reducible to

$$\frac{X}{\alpha + \beta} = \frac{Y}{\beta - \alpha} = \frac{1 - Z}{2}$$

For all values of α and β these lines pass through the point $(0, 0, 1)$. Hence a point of the plane corresponding to a given point P upon the sphere is the point of intersection with the plane of the line joining P with the pole $(0, 0, 1)$. This form of representation is called the *stereographic projection* of the sphere upon the plane. It is evident that a line in the plane corresponds to a circle on the sphere, this circle is determined by the plane of the pole and the given line.

We will close this chapter with a few remarks about the conformal representation of the sphere upon itself. From the foregoing results we know that every such representation of the sphere (103) is given by equations of similar form in α_1, β_1 , where the latter are given by (86) or (87), and that for conformal representations with circles in correspondence α_1 and β_1 have the values (97) or (98)

We consider in particular the case

$$(105) \quad \alpha_1 = \frac{a_1\alpha + a}{\alpha_3\alpha + a_4}, \quad \beta_1 = \frac{a_4\beta - a_3}{-a\beta + a_1}$$

The expressions of the linear elements of the sphere are found to be reducible to

$$ds' = \frac{4 \, d\alpha \, d\beta}{(1 + \alpha\beta)}, \quad ds_1^2 = \frac{4 \, d\alpha_1 \, d\beta_1}{(1 + \alpha_1\beta_1)} = \frac{4 \, d\alpha \, d\beta}{(1 + \alpha\beta)^2}$$

* Here we have used the upper signs in (104)

equations (105) define an isometric representation of the sphere upon itself. Since angles are preserved in the same sense, this representation may be looked upon as determining a new configuration upon the sphere into new coordinates.

The stationary points in the general motion, if there are any, correspond to values of α and β , which are roots of the following equations

$$a_2 t^2 + (a_4 - a_1)t - a_3 = 0, \quad a_3 t^2 + (a_4 - a_1)t - a_2 = 0.$$

If t_1 are the roots of the former, those of the latter are $-1/t_1$. Hence there are four points stationary in the motion, whose spherical coordinates are

$$\left(t_1, -\frac{1}{t_1}\right), \left(t_2, -\frac{1}{t_2}\right), \left(t_3, -\frac{1}{t_3}\right), \left(t_4, -\frac{1}{t_4}\right).$$

From (103) it is seen that the first two are at infinity, and the other two determine points on the sphere, so that the motion is a rotation about these points. If the s -axis is taken for the axis of rotation, we have from (103) that the roots of (106) must be ∞ and 0 , so that (105) becomes

$$\alpha_1 = \frac{a_1}{a_4} \alpha, \quad \beta_1 = \frac{a_2}{a_4} \beta$$

If the rotation is real, these equations must be of the form

$$\alpha_1 = e^{i\theta} \alpha, \quad \beta_1 = e^{-i\theta} \beta,$$

where θ is the angle of rotation.

EXAMPLES

1. Find the equations of the surface of revolution with the linear element $ds^2 = (a^2 - u^2)dv^2 + du^2$.

2. Find the loxodromic curves on the surface

$$x = u \cos v, \quad y = u \sin v, \quad z = a \cosh^{-1} \frac{u}{a}$$

3. Find the equations of the surface when referred to an orthogonal system of coordinates.

4. Find the general equations of the conformal representation of the sphere upon the plane.

5. Show that for the surface generated by the revolution of the sphere of radius a about the base of the latter the linear element is reducible to $ds^2 = a^2 d\theta^2 + u dv^2$.

5. A great circle on the unit sphere cuts the meridian $v = 0$ in latitude α under angle α . Find the equation of its stereographic projection

6. Determine the stereographic projection of the curve $x = a \sin u \cos u$, $y = a \cos^2 u$, $z = a \sin u$ from the pole $(0, a, 0)$

GENERAL EXAMPLES

1. When there is a one-to-one point correspondence between two surfaces, the cross-ratio of four tangents to one surface at a point is equal to the cross-ratio of the corresponding tangents to the other

2. Given the paraboloid

$$x = 2au \cos v, \quad y = 2bu \sin v, \quad z = 2u^2(a \cos^2 v + b \sin^2 v),$$

where a and b are constants. Determine the equation of the curves on the surface, such that the tangent planes along a curve make a constant angle with the xy -plane. Show that the generators of the developable Σ , enveloped by these planes, make a constant angle with the x -axis, and express the coordinates of the edge of regression in terms of u .

3. Find the orthogonal trajectories of the generators of the surface Σ in Ex 2. Show that they are plane curves and that their projections on the xy -plane are involutes of the projection of the edge of regression

4. Let C be a curve on a cone of revolution which cuts the generators under constant angle, and C_1 the locus of the centers of curvature of C . Show that C_1 lies upon a cone whose elements it cuts under constant angle

5. If a developable of a curve is developed upon a plane, the curve is rectifying.

6. If a rectifying developable of a curve is developed upon a plane, the surface is a straight line

7. Determine $\phi(v)$ so that the right conoid,

$$x = u \cos v, \quad y = u \sin v, \quad z = \phi(v),$$

shall be applicable to a surface of revolution

8. Determine the equations of a conformal representation of the plane upon itself for which the parallels to the axes in the α_1 plane correspond to lines through a point (a, b) and circles concentric about it in the α plane

9. The equation $\alpha_1 = c \sin \alpha$, where c is a constant, defines a conformal representation of the plane upon itself such that the lines parallel to the axes in the α plane correspond to confocal ellipses and hyperbolas in the α_1 plane

10. In the conformal representation of the plane upon itself, given by $\alpha_1 = \alpha^2$, to lines parallel to the axes in the α_1 plane there correspond equilateral hyperbolas in the α -plane, and to the pencil of rays through a point in the α_1 -plane and the circles concentric about it there corresponds a system of equilateral hyperbolas through the corresponding point in the α -plane and a family of confocal Cassini ovals

11. When the sides of a triangle upon a surface of revolution are loxodromic curves, the sum of the three angles is equal to two right angles

12. The only conformal perspective representation of a sphere upon a plane is given by (104)

Now that equations (105) and the equations obtained from (105) by the replacement of α and β define the most general isometric representation of the surface on itself

Let each of two surfaces S, S_1 be defined in terms of parameters u, v , and on each with the same values of the parameters correspond. If $H = H_1$, the latter is the function for S_1 analogous to H for S , corresponding elements are equal and the representation is said to be *equivalent*.* If $H \neq H_1$ and the elements of S are changed in accordance with the equations $u' = \phi(u, v)$, $v' = \psi(u, v)$, the condition that the equations $u' = u$, $v' = v$ define an equivalent representation of S and S_1 is

$$\frac{\partial \phi}{\partial u} \frac{\partial \psi}{\partial v} - \frac{\partial \phi}{\partial v} \frac{\partial \psi}{\partial u} = \frac{H(u, v)}{H_1(\phi, \psi)}$$

Under what conditions do the equations

$$x' = a_1x + a_2y + a_3, \quad y' = b_1x + b_2y + b_3$$

represent an equivalent representation of the plane upon itself?

Now that the equations

$$x = \int r \sqrt{1 + \phi^2} dr, \quad y = v,$$

represent an equivalent representation of the surface of revolution (99) upon the

surface of a sphere and circumscribed circular cylinder. If the points at which the circles perpendicular to the axis of the latter meet the two surfaces correspond, the representation is equivalent.

Find an equivalent representation of the sphere upon the plane such that the great circles correspond to lines parallel to the y axis and the meridians to lines for which the extremities of one of the principal axes are $(a, 0)$, $(-a, 0)$.

* German writers call it "flächentreu."

CHAPTER IV

GEOMETRY OF A SURFACE IN THE NEIGHBORHOOD OF A POINT

§25. **Fundamental coefficients of the second order** In this chapter we study the form of a surface in the neighborhood of a point M of it, and the character of the curves which lie upon the surface and pass through the point. We recall that the tangents at M to all these curves lie in a plane, — the *tangent plane* to the surface at the point.

The equation of the tangent plane at $M(x, y, z)$, namely (II, 11), may be written

$$(1) \quad (\xi - x)X + (\eta - y)Y + (\zeta - z)Z = 0,$$

where we have put

$$X = \frac{1}{H} \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}, \quad Y = \frac{1}{H} \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix}, \quad Z = \frac{1}{H} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

We define the positive direction of the normal (§ 25) to be that for which the functions X, Y, Z are the direction cosines. From this definition it follows that the tangents to the curves $v = \text{const}$ and $u = \text{const}$. at a point and the normal at the point have the same mutual orientation as the x -, y -, and z axes

From (2) follow the identities

$$(3) \quad \sum X \frac{\partial x}{\partial u} = 0, \quad \sum X \frac{\partial x}{\partial v} = 0,$$

which express the fact that the normal is perpendicular to the tangents to the coordinate curves. In consequence of these identities the expression for the distance p from a point $M'(u + du, v + dv)$ to the tangent plane at M is of the second order in du and dv . It may be written

$$(4) \quad p = \sum X dx = \frac{1}{2} (D du^2 + 2D' dudv + D'' dv^2) + e,$$

ϵ denotes the aggregate of terms of the third and higher order in du and dv , and the functions D , D' , D'' are defined by

$$D = \sum X \frac{\partial^2 x}{\partial u^2}, \quad D' = \sum X \frac{\partial^2 x}{\partial u \partial v}, \quad D'' = \sum X \frac{\partial^2 x}{\partial v^2}$$

relations (3) be differentiated with respect to u and v respectively, we get

$$\begin{cases} \sum X \frac{\partial^2 x}{\partial u^2} + \sum \frac{\partial X}{\partial u} \frac{\partial x}{\partial u} = 0, & \sum X \frac{\partial^2 x}{\partial u \partial v} + \sum \frac{\partial X}{\partial v} \frac{\partial x}{\partial u} = 0, \\ \sum X \frac{\partial^2 x}{\partial u \partial v} + \sum \frac{\partial X}{\partial u} \frac{\partial x}{\partial v} = 0, & \sum X \frac{\partial^2 x}{\partial v^2} + \sum \frac{\partial X}{\partial v} \frac{\partial x}{\partial v} = 0 \end{cases}$$

so equations (5) may be written

$$\begin{cases} D = \sum X \frac{\partial^2 x}{\partial u^2} = - \sum \frac{\partial X}{\partial u} \frac{\partial x}{\partial u}, \\ D' = \sum X \frac{\partial^2 x}{\partial u \partial v} = - \sum \frac{\partial X}{\partial u} \frac{\partial x}{\partial v} = - \sum \frac{\partial X}{\partial v} \frac{\partial x}{\partial u}, \\ D'' = \sum X \frac{\partial^2 x}{\partial v^2} = - \sum \frac{\partial X}{\partial v} \frac{\partial x}{\partial v} \end{cases}$$

the quadratic differential form

$$\Phi = D du^2 + 2 D' dudv + D'' dv^2$$

is called the *second fundamental form* of the surface, and the functions D , D' , D'' the *fundamental coefficients of the second order*. We leave it to the reader to show that these coefficients, like those of the first order, are invariant for any displacement of the surface in space. Hereafter we shall have occasion to use two sets of formulas which may now be derived.

From the equations of definition,

$$E = \sum \left(\frac{\partial x}{\partial u} \right)^2, \quad F = \sum \frac{\partial x}{\partial u} \frac{\partial x}{\partial v}, \quad G = \sum \left(\frac{\partial x}{\partial v} \right)^2,$$

and, by differentiation and simple reduction, the following

$$\begin{cases} \sum \frac{\partial^2 x}{\partial u^2} \frac{\partial x}{\partial u} = \frac{1}{2} \frac{\partial E}{\partial u}, & \sum \frac{\partial^2 x}{\partial u \partial v} \frac{\partial x}{\partial u} = \frac{cF}{c u} - \frac{1}{2} \frac{cE}{c v}, \\ \sum \frac{\partial^2 x}{\partial u \partial v} \frac{\partial x}{\partial v} = \frac{1}{2} \frac{\partial F}{\partial v}, & \sum \frac{\partial^2 x}{\partial v^2} \frac{\partial x}{\partial v} = \frac{1}{2} \frac{cG}{c v}, \\ \sum \frac{\partial^2 x}{\partial u^2} \frac{\partial x}{\partial v} = \frac{\partial F}{\partial u} - \frac{1}{2} \frac{cG}{c u}, & \sum \frac{\partial^2 x}{\partial v^2} \frac{\partial x}{\partial u} = \frac{1}{2} \frac{\partial G}{\partial v} \end{cases}$$

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Again, if the expressions (9) be substituted in the left-hand members of the following equations, the reduced results may be written by means of (2) in the form indicated

$$(11) \quad \begin{cases} E \frac{\partial x}{\partial v} - F \frac{\partial x}{\partial u} = H \left(Y \frac{\partial z}{\partial u} - Z \frac{\partial y}{\partial u} \right), \\ F \frac{\partial x}{\partial v} - G \frac{\partial x}{\partial u} = H \left(Y \frac{\partial z}{\partial v} - Z \frac{\partial y}{\partial v} \right) \end{cases}$$

Similar identities can be found by permuting the letters x, y, z , E, F, G .

From the fundamental relation

$$X^2 + Y^2 + Z^2 = 1$$

we obtain, by differentiation with respect to u and v respectively, the identities

$$(12) \quad \sum X \frac{\partial X}{\partial u} = 0, \quad \sum X \frac{\partial X}{\partial v} = 0$$

and (7) constitute a system of three equations in $\frac{\partial X}{\partial u}, \frac{\partial Y}{\partial u}, \frac{\partial Z}{\partial u}$, and a system linear in $\frac{\partial X}{\partial v}, \frac{\partial Y}{\partial v}, \frac{\partial Z}{\partial v}$. Solving for $\frac{\partial X}{\partial u}$ and for $\frac{\partial X}{\partial v}$, we find, by means of (11),

$$(13) \quad \begin{cases} \frac{\partial X}{\partial u} = \frac{FD' - GD}{H^2} \frac{\partial x}{\partial u} + \frac{FD - ED'}{H^2} \frac{\partial x}{\partial v}, \\ \frac{\partial X}{\partial v} = \frac{FD'' - GD'}{H^2} \frac{\partial x}{\partial u} + \frac{FD' - ED''}{H^2} \frac{\partial x}{\partial v} \end{cases}$$

The expressions for $\frac{\partial Y}{\partial u}, \frac{\partial Z}{\partial v}$ are obtained by replacing x by y and z respectively

By means of these equations we shall prove that a real surface whose first and second fundamental coefficients are in proportion, thus

$$(14) \quad \frac{D}{E} = \frac{D'}{F} = \frac{D}{G} = -\lambda$$

where λ denotes the factor of proportionality, is a sphere or a plane. We assume that the minimal lines are parametric. In consequence we have

$$E = G = D = D' = 0,$$

so that equations (13) become

$$(15) \quad \frac{\partial X}{\partial u} = \lambda \frac{\partial x}{\partial u} \quad \frac{\partial X}{\partial v} = \lambda \frac{\partial x}{\partial v}$$

action λ must satisfy the condition

$$\frac{\partial}{\partial v} \left(\lambda \frac{\partial x}{\partial u} \right) = \frac{\partial}{\partial u} \left(\lambda \frac{\partial x}{\partial v} \right),$$

reduces to $\frac{\partial \lambda}{\partial v} \frac{\partial x}{\partial u} - \frac{\partial \lambda}{\partial u} \frac{\partial x}{\partial v} = 0$. Moreover, we have two other equations of the same form, obtained from the above by replacing x by y and z respectively. Since λ is a constant, we get

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial u} \frac{\partial z}{\partial u} = \frac{\partial x}{\partial v} \frac{\partial y}{\partial v} \frac{\partial z}{\partial v}$$

possible for a real surface, we must have $\frac{\partial \lambda}{\partial u} = \frac{\partial \lambda}{\partial v} = 0$, that is, λ is a constant.

When λ is zero the functions X, Y, Z given by (15) are constant, and the surface is a plane. When λ is any other constant, we get, from (15),

$$X = \lambda x + a, \quad Y = \lambda y + b, \quad Z = \lambda z + c,$$

a, b, c are constants. From these equations we obtain $(\lambda x + a)^2 + (\lambda y + b)^2 + (\lambda z + c)^2 = 1$. Since this is the general equation of a sphere, it follows that the condition is necessary as well as sufficient.

Radius of normal curvature. Consider on a surface S any curve C through a point M . The direction of its tangent, MT , is determined by a value of dv/du . Let $\bar{\omega}$ denote the angle which the positive direction of the normal to the surface makes with the positive direction of the principal normal to C at M , angles being measured toward the positive binormal. If we use the notation of the last chapter, and take the arc of C for its parameter, we have

$$\cos \bar{\omega} = Xl + Ym + Zn = \rho \left(X \frac{d^2x}{ds^2} + Y \frac{d^2y}{ds^2} + Z \frac{d^2z}{ds^2} \right)$$

where $\frac{du}{ds}$ and $\frac{dv}{ds}$ the derivatives in the parenthesis have the same values as in the last chapter.

$$\frac{d^2x}{ds^2} = \frac{\partial x}{\partial u} \left(\frac{du}{ds} \right)' + 2 \frac{\partial x}{\partial u \partial v} \frac{du}{ds} \frac{dv}{ds} + \frac{\partial^2 x}{\partial v^2} \left(\frac{dv}{ds} \right)^2 + \frac{\partial x}{\partial u} \frac{d^2u}{ds^2} + \frac{\partial x}{\partial v} \frac{d^2v}{ds^2}$$

the above equation is equivalent to

$$\cos \bar{\omega} = \frac{D du' + 2 D' du dv + D'' dv'}{\rho (L du^2 + 2 F du dv + G dv^2)}$$

The right-hand member of this equation depends only upon curvilinear coordinates of the point and the direction of MT , the same for all curves with this tangent at M . Since ρ is positive, the angle $\bar{\omega}$ cannot be greater than a right angle for one tangent to MT , if it is less than a right angle for any other

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curve tangent to MT , and vice versa. We consider in particular the curve in which the surface is cut by the plane determined by MT and the normal to the surface at M . We call it the *normal section tangent to MT* , and let ρ_n denote its radius. Since the right-hand member of equation (16) is the same for C and the normal section tangent to it, we have

$$(17) \quad \frac{\cos \bar{\omega}}{\rho} = \frac{e}{\rho_n},$$

where e is $+1$ or -1 , according as $\bar{\omega}$ is less or greater than a right angle; let ρ and ρ_n are positive. Equation (17) gives the following theorem of Meusnier

The center of curvature of any curve upon a surface is the projection upon its osculating plane of the center of curvature of the normal section tangent to the curve at the point

In order to avoid the ambiguous sign in (17), we introduce a new function R which is equal to ρ_n when $0 \leq \bar{\omega} \leq \pi/2$, and to $-\rho_n$ when $\pi/2 \leq \bar{\omega} \leq \pi$, and call it the *radius of normal curvature* of the surface for the given direction MT . As thus defined, R is given by

$$(18) \quad \frac{1}{R} = \frac{Ddu^2 + 2D'dudv + D''dv^2}{Edu^2 + 2Fdudv + Gdv^2}$$

Now we may state Meusnier's theorem as follows

If a segment, equal to twice the radius of normal curvature for a given direction at a point on a surface, be laid off from the point on the normal to the surface, and a sphere be described with the segment for diameter, the circle in which the sphere is met by the osculating plane of a curve with the given direction at the point is the circle of curvature of the curve

50 Principal radii of normal curvature. If we put $t = \frac{dv}{du}$, equation (18) becomes

$$(19) \quad \frac{1}{R} = \frac{D + 2D't + D''t^2}{E + 2Ft + Gt^2}$$

When the proportion (14) is satisfied, R is the same for all values of t , being ∞ for the plane, and the constant $-1/\lambda$ for the sphere. For any other surface R varies continuously with t . And so we

3 values of t for which R is a maximum or minimum
 end we differentiate the above expression with respect
 1 put the result equal to zero This gives

$$D' + D''t)(E + 2Ft + Gt^2) - (F + Gt)(D + 2D't + D''t^2) = 0,$$

$$(FD'' - GD')t^2 + (ED'' - GD)t + (ED' - FD) = 0$$

t any loss of generality we can assume that the parametric
 ue such that $E \neq 0$, so that we have the identity

$$\begin{aligned} & ED'' - GD)^2 - 4(FD'' - D'G)(ED' - FD) \\ &= 4 \frac{H^2}{E^4} (ED' - FD)^2 + \left[ED'' - GD - \frac{2F}{E} (ED' - FD) \right]^2 \end{aligned}$$

he surface is real, and the parameters also, the right-hand
 of this equation is positive Since the left-hand member
 is discriminant of equation (21), the latter has two real and
 roots * When the test (III, 34) is applied to equation (21),
 and that the two directions at a point determined by the
 (21) are perpendicular Hence

*every ordinary point of a surface there is a direction for which
 us of normal curvature is a maximum and a direction for
 is a minimum, and they are at right angles to one another*

3 limiting values of R are called the *principal radii of
 curvature* at the point They are equal to each other for
 ie and the sphere, and these are the only real surfaces
 is property

(20) and (19) we have

$$\frac{D' + D''t}{F + Gt} = \frac{D + D't}{E + Ft} = \frac{1}{R}$$

he following relations hold between the principal radii and
 esponding values of t

$$\begin{cases} E + Ft - R(D + D't) = 0, \\ F + Gt - R(D' + D''t) = 0 \end{cases}$$

let that the two roots be equal the discriminant must vanish This is impos-
 al surfaces other than spheres and planes as seen from (22) For an imaginary
 this kind referred to its lines of length zero we have from (21) that D or D'
 ce $F \neq 0$ The vanishing of the discriminant is also the necessary and sufficient
 hat the numerator and denominator in (19) have a common factor

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When t is eliminated from these equations, we get the equation

$$(24) \quad (DD'' - D'^2)E^2 - (ED'' + GD - 2FD')R + (EG - F^2) = 0,$$

whose roots are the principal radii. If these roots be denoted by ρ_1 and ρ_2 , we have

$$(25) \quad \begin{cases} \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{ED'' + GD - 2FD'}{H^2}, \\ \frac{1}{\rho_1 \rho_2} = \frac{DD'' - D'^2}{H^2} \end{cases}$$

Although equations (14) hold at all points of a sphere and a plane, and for no other surface, it may happen that for certain particular points of a surface they are satisfied. At such points R , as given by (19), is the same for all directions, and the equation (21) vanishes identically. When points of this kind exist they are called *umbilical points* of the surface.

EXAMPLES

- 1 When the equation of the surface is $z = f(x, y)$, show that

$$X, Y, Z = \frac{-p, -q, 1}{\sqrt{1+p^2+q^2}},$$

$$D, D', D'' = \frac{r, s, t}{\sqrt{1+p^2+q^2}},$$

where $p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}$

- 2 Show that the normals to the right conoid

$$x = u \cos v, \quad y = u \sin v, \quad z = \phi(v)$$

along a generator form a hyperbolic paraboloid

- 3 Show that the principal radii of normal curvature of a right conoid at a point differ in sign

- 4 Find the expression for the radius of normal curvature of a surface of revolution at a point in the direction of the loxodromic curve through it, which makes the angle α with the meridians

- 5 Show that the meridians and parallels on a surface of revolution, $x = u \cos v, y = u \sin v, z = \phi(u)$, are the directions in which the radius of normal curvature is maximum and minimum, that the principal radii are given by

$$\frac{1}{\rho_1} = \frac{\phi''(u)}{(1+\phi'^2)^{\frac{3}{2}}}, \quad \frac{1}{\rho_2} = \frac{\phi'(u)}{u(1+\phi'^2)^{\frac{1}{2}}},$$

and that ρ_2 is the segment of the normal between the point of the surface and the intersection of the normal with the z axis

- 6 Show that $\Delta_1 x = 1 - X^2$ and $\Delta_1(x, y) = -XY$, where the differential parameters are formed with respect to the linear element of the surface

lines of curvature Equations of Rodrigues. We have seen curves defined by equation (21), written

$$ED' - FD) du^2 + (ED'' - GD) du dv + (FD'' - GD') dv^2 = 0,$$

orthogonal system As defined, the two curves of the system through a point on the surface determine the directions at the point which the radii of normal curvature have their maximum

minimum values These curves are called *lines of curvature*, and their tangents at the point are the *principal directions* for the point. We shall discuss another geometric property which we shall now find

normals to a surface along a curve ruled surface In order that the surface be developable, the normals must be tangent to a curve (§ 27), as in fig 12 If the coordinates of a point M_1 on the normal at M be denoted by x_1, y_1, z_1 , we have

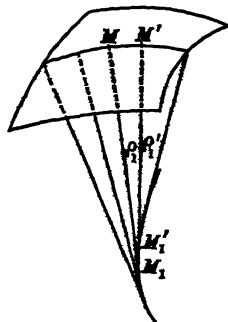


FIG 12

$$x_1 = x + rX, \quad y_1 = y + rY, \quad z_1 = z + rZ,$$

where r denotes the length MM_1 . If M_1 be a point of the edge of the surface, we must have

$$\frac{r + r dX + X dr}{X} = \frac{dy + r dY + Y dr}{Y} = \frac{dz + r dZ + Z dr}{Z}$$

Equating the numerators and the denominators of the respective fractions by X, Y, Z , and combining, we find that the common denominator is dr . Hence the above equations reduce to

$$dx + r dX = 0, \quad dy + r dY = 0, \quad dz + r dZ = 0,$$

when the parametric coordinates are used,

$$\begin{cases} \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + r \left(\frac{\partial X}{\partial u} du + \frac{\partial X}{\partial v} dv \right) = 0, \\ \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv + r \left(\frac{\partial Y}{\partial u} du + \frac{\partial Y}{\partial v} dv \right) = 0, \\ \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv + r \left(\frac{\partial Z}{\partial u} du + \frac{\partial Z}{\partial v} dv \right) = 0 \end{cases}$$

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If these equations be multiplied by $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}$ respectively and added, and by $\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}$ respectively and added, we get

$$E du + F dv - r(D du + D' dv) = 0,$$

$$F du + G dv - r(D' du + D'' dv) = 0$$

But these equations are the same as (23). Hence

The normals to a surface along a curve of it form a ruled surface which is a developable only when the curve is a line of curvature, in this case the points of the edge of regression are the centers of normal curvature of the surface in the direction of the curve

The coordinates of the principal centers of curvature are

$$(25) \quad \begin{cases} x_1 = x + \rho_1 X, & y_1 = y + \rho_1 Y, & z_1 = z + \rho_1 Z, \\ x_2 = x + \rho_2 X, & y_2 = y + \rho_2 Y, & z_2 = z + \rho_2 Z \end{cases}$$

When the parametric curves are the lines of curvature, equation (26) is necessarily of the form

$$(29) \quad \lambda du dv = 0,$$

and consequently we must have $ED' - FD = 0$, $FD'' - GD' = 0$

Since $ED'' - GD \neq 0$, these equations are equivalent to

$$(30) \quad F = 0, \quad D' = 0$$

Conversely, when these conditions are satisfied equation (26) reduces to the form (29). Hence

A necessary and sufficient condition that the lines of curvature parametric is that F and D' be zero

Let the lines of curvature be parametric, and let ρ_1 and ρ denote the principal radii of normal curvature of the surface in the directions of the lines of curvature $v = \text{const}$ and $u = \text{const}$ respectively. From (19) we find

$$(31) \quad \frac{1}{\rho_1} = \frac{D}{E}, \quad \frac{1}{\rho} = \frac{D''}{G},$$

and equations (13) become

$$(32) \quad \begin{cases} \frac{\partial x}{\partial u} = -\rho_1 \frac{\partial X}{\partial u}, & \frac{\partial y}{\partial u} = -\rho_1 \frac{\partial Y}{\partial u}, & \frac{\partial z}{\partial u} = -\rho_1 \frac{\partial Z}{\partial u}, \\ \frac{\partial x}{\partial v} = -\rho_2 \frac{\partial X}{\partial v}, & \frac{\partial y}{\partial v} = -\rho_2 \frac{\partial Y}{\partial v}, & \frac{\partial z}{\partial v} = -\rho_2 \frac{\partial Z}{\partial v} \end{cases}$$

These equations are called the *equations of Rodrigues*

2. Total and mean curvature. Of fundamental importance in discussion of the nature of a surface in the neighborhood of a point are the product and the sum of the principal curvatures at that point. They are called the *total curvature* * of the surface at that point and the *mean curvature* respectively. If they be denoted K and K_m , we have, from (25),

$$\begin{cases} K = \frac{1}{\rho_1 \rho_2} = \frac{DD'' - D'^2}{H^2}, \\ K_m = \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{ED'' + GD - 2FD'}{H^2} \end{cases}$$

When K is positive at a point M , the two principal radii have same sign, and consequently the two centers of principal curvature lie on the same side of the tangent plane. As all the centers of curvature of other normal sections lie between these two, the position of the surface in the neighborhood of M lies entirely on one side of the tangent plane. This can be seen also in another way.

Since H^2 is positive, we must have $DD'' - D'^2 > 0$. Hence the distance from a near-by point to the tangent plane at M , since it is proportional to the fundamental form Φ (§ 48), does not change sign as dv/du is varied.

When K is negative at M , the principal radii differ in sign, and consequently part of the surface lies on one side of the tangent plane and part on the other. In particular there are two directions, in which

$$D du^2 + 2D' dudv + D'' dv^2 = 0,$$

which the normal curvature is zero. In these directions the distances of the near-by points of the surface from the tangent plane, given by (4), are quantities of the third order at least. Hence these lines are the tangents at M to the curve in which the tangent plane at M meets the surface.

At the points for which K is zero, one of the principal radii is infinite. At these points Φ has the form $(\sqrt{D} du + \sqrt{D''} dv)$ and vanishes in the direction $\sqrt{D} du + \sqrt{D''} dv = 0$. But as dv/du passes through the value given by this equation, Φ does not change sign, since the surface lies on one side of the tangent plane and is tangent to it along the above direction.

* The total curvature is sometimes called the *Gaussian curvature* after the celebrated mathematician who suggested it as a suitable measure of the curvature at a point. Cf. Gauss, p. 15.

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An *anulus*, ring, or *tore*, is a surface with points of all three kinds. Such a surface may be generated by the rotation of a circle of radius a about an axis in the plane of the circle and at a distance b ($> a$) from the center of the circle. The points at the distance b from the axis lie in two circles, and the tangent plane to the *tore* at a point of either of the circles is tangent all along the circle. Hence the surface has zero curvature at all points of these circles. At every point whose distance from the axis is greater than b the surface lies on one side of the tangent plane, whereas, when the distance is less than b , the tangent plane cuts the surface.

There are surfaces for which K is positive at every point, as, for example, the ellipsoid and the elliptic paraboloid. Moreover, for the hyperboloid of one sheet and the hyperbolic paraboloid the curvature is negative at every point. Surfaces of the former type are called *surfaces of positive curvature*, of the latter type *surfaces of negative curvature*.

Later (§ 64) we shall prove that when K is zero at all points of a surface the latter is developable, and conversely.

53. Equation of Euler Dupin indicatrix. When the lines of curvature are parametric, equation (18) can be written, in consequence of (III, 23) and (31), in the form

$$(34) \quad \frac{1}{R} = \frac{\cos^2 \theta}{\rho_1} + \frac{\sin^2 \theta}{\rho_2},$$

where θ is the angle between the directions whose radii of normal curvature are R and ρ_1 . Equation (34) is called the *equation of Euler*

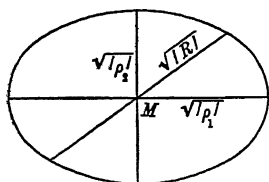


FIG 13

When the total curvature K at a point is positive, ρ_1 and ρ_2 for the point have the same sign, and R has this sign for all directions. If the tangents to the lines of curvature at the point M be taken for coordinate axes, with respect to which ξ and η are coordinates, and segments of

length $\pm \sqrt{|R|}$ be laid off from M in the two directions corresponding to R , the locus of the end points of these segments is the ellipse (fig 13) whose equation is

$$\frac{\xi^2}{[\rho_1]} + \frac{\eta^2}{[\rho_2]} = 1$$

This ellipse is called the *Dupin indicatrix* for the point. When, in particular, ρ_1 and ρ_2 are equal, the indicatrix is a circle. Hence the Dupin indicatrix at an umbilical point is a circle (§ 50). For this reason such a point is sometimes called a *circular point*.

If K is negative ρ_1 and ρ_2 differ in sign, and consequently values of R are positive and the others are negative. In directions for which R is positive we lay off the segments and in the other directions $\pm\sqrt{-R}$. The locus of the ends of these segments contains the conjugate hyperbolas whose equations are

$$\frac{\eta^2}{\rho_2} = 1, \quad \frac{\xi^2}{\rho_1} + \frac{\eta^2}{\rho_2} = -1$$

remark that R is infinite for directions given by

$$\tan^2 \theta = -\frac{\rho_2}{\rho_1},$$

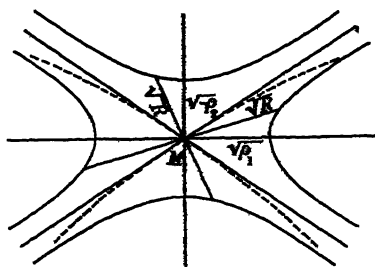


FIG 14

In other words, in the directions of the asymptotes to the hyperbolas. The above locus is the *Dupin indicatrix* for the point M . If, when $K = 0$ the equation of the indicatrix is of one of the following forms

$$\xi^2 = |\rho_1|, \quad \eta^2 = |\rho_2|,$$

it is a pair of parallel straight lines. In view of the foregoing a point of a surface is called *elliptic*, *hyperbolic*, or *parabolic*, according as the total curvature at the point is positive, negative, or zero.

In consequence of (4) the expression for the distance p upon the plane to a surface at a point M from a near-by point P of the surface is given by

$$\frac{E du^2}{\rho_1} + \frac{G dv^2}{\rho_2} = 2p,$$

in terms of higher order. But $\sqrt{E} du$ and $\sqrt{G} dv$ are the distances, to within terms of higher order, of P from the normal to the surface at M in the directions of the lines of curvature. Hence the plane parallel to the tangent plane and at a distance p from it cuts the surface in the curve

$$\frac{\xi^2}{\rho_1} + \frac{\eta^2}{\rho_2} = 2p, \quad \zeta = p$$

which is a conic similar to the Dupin indicatrix at an elliptic or parabolic point, and to a part of the indicatrix at a hyperbolic point.

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EXAMPLES

1. Show that the meridians and parallels of a surface of revolution are its lines of curvature, and determine the character of the developable surfaces formed by the tangents to the surface along these lines.

2. Show that the parametric lines on the surface

$$x = \frac{a}{2}(u+v), \quad y = \frac{b}{2}(u-v), \quad z = \frac{uv}{2},$$

are straight lines. Find the lines of curvature

3. When a surface is defined by $z = f(x, y)$, the expressions for the curvatures are

$$K = \frac{rt - s^2}{(1 + p^2 + q^2)^2}, \quad K_m = \frac{(1 + p^2)t + (1 + q^2)r - 2pqs}{(1 + p^2 + q^2)^{\frac{3}{2}}},$$

and the equation of the lines of curvature is

$$[(1 + p^2)s - pqr]dx^2 + [(1 + p^2)t - (1 + q^2)r]dxdy + [pqt - (1 + q^2)s]dy^2 = 0$$

4. The principal radii of the surface $y \cos \frac{z}{a} - x \sin \frac{z}{a} = 0$ at a point (x, y, z) are equal to $\pm \frac{x^2 + y^2 + a^2}{a}$. Find the lines of curvature

5. Derive the equations of the tore, defined in § 52, and prove therefrom the results stated

6. The sum of the normal curvatures in two orthogonal directions is constant

7. The Euler equation can be written

$$R = \frac{2\rho_1\rho_2}{\rho_1 + \rho_2 - (\rho_1 - \rho_2)\cos 2\theta}$$

54 Conjugate directions at a point **Conjugate systems** Two curves on a surface through a point M are said to have *conjugate directions* when their tangents at M coincide with conjugate diameters of the Dupin indicatrix for the point. These tangents are also parallel to conjugate diameters of the conic in which the surface is cut by a plane parallel to the tangent plane to M and very near it. Let P denote a point of this conic and N the point in which its plane α cuts the normal at M . The tangent plane to the surface at P meets the plane α in the tangent line at P to the conic. Moreover, this tangent line is parallel to the diameter conjugate to NP . Hence as P approaches M this tangent line approaches the diameter of the Dupin indicatrix, which is conjugate to the diameter in the direction MP . Hence we have (cf § 27)

The characteristic of the tangent plane to a surface, as the point of contact moves along a curve, is the tangent to the surface in the direction conjugate to the curve

if this theorem we derive the analytical condition for sections

tion of the tangent plane is

$$(\xi - x)X + (\eta - y)Y + (\zeta - z)Z = 0,$$

current coordinates, the characteristic is defined by and

$$(\xi - x) \frac{dX}{ds} + (\eta - y) \frac{dY}{ds} + (\zeta - z) \frac{dZ}{ds} = 0,$$

is arc of the curve along which the point of contact $\delta y, \delta z$ denote increments of x, y, z in the direction the curve, we have, from the above equations,

$$\delta x dX + \delta y dY + \delta z dZ = 0$$

of u and v in the conjugate direction be denoted by s equation may be written

$$D du \delta u + D' (du \delta v + dv \delta u) + D'' dv \delta v = 0$$

?

ons conjugate to any curve of the family

$$\phi(u, v) = \text{const}$$

$$D \frac{\partial \phi}{\partial v} - D' \frac{\partial \phi}{\partial u} \delta u + \left(D' \frac{\partial \phi}{\partial v} - D'' \frac{\partial \phi}{\partial u} \right) \delta v = 0$$

ifferential equation of the first order and first degree, the parameter family of curves. These curves and the first are said to form a *conjugate system*. Moreover, families of curves are said to form a *conjugate system* if every tangent to a curve of each family at their point of intersection is conjugate directions.

it follows that the curves conjugate to the curves defined by $D \delta u + D' \delta v = 0$. Consequently, in order that the curves $u = \text{const}$, we must have D' equal to zero. If the same is also true, we have

the necessary and sufficient condition that the parametric curves form a conjugate system is that D' be zero

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We have seen (§ 51) that the lines of curvature are characterized by the property that, when they are parametric, the coefficients F and D' are zero. Hence

The lines of curvature form a conjugate system and the only orthogonal conjugate system.

If the lines of curvature are parametric, and the angles which a pair of conjugate directions make with the tangent to the curve $u = \text{const.}$ are denoted by θ and θ' , we have

$$\tan \theta = \sqrt{\frac{G}{E}} \frac{dv}{du}, \quad \tan \theta' = \sqrt{\frac{G}{E}} \frac{\delta v}{\delta u},$$

so that equation (36) may be put in the form

$$(39) \quad \tan \theta \tan \theta' = -\frac{\rho_2}{\rho_1},$$

which is the well-known equation of conjugate directions of a conic

55. Asymptotic lines Characteristic lines When θ' is equal to θ , equation (39) reduces to (35). Hence the *asymptotic directions* are self-conjugate. If in equation (36) we put $\delta v/\delta u = dv/du$, we obtain

$$(40) \quad D du^2 + 2 D' du dv + D'' dv^2 = 0,$$

which determines, consequently, the asymptotic directions at each point of the surface. This equation defines a double family of curves upon the surface, two of which pass through each point and admit as tangents the asymptotic directions at the point. They are called the *asymptotic lines* of the surface.

The asymptotic lines are imaginary on surfaces of positive curvature, real on surfaces of negative curvature, and consist of a single real family on a surface of zero curvature.

Recalling the results of § 52, we say that the tangent plane to a surface at a point cuts the surface in asymptotic lines in the neighborhood of the point. As an immediate consequence, we have that the generators of a ruled surface form one family of asymptotic lines.

Since an asymptotic line is self-conjugate, the characteristics of the tangent plane as the point of contact moves along an asymptotic line are the tangents to the latter. Hence the osculating plane of an asymptotic line at a point is the tangent plane to the

at the point, and consequently the asymptotic line is the regression of the developable circumscribing the surface is asymptotic line. This follows also from equation (16) (40) we have the theorem

necessary and sufficient condition that the asymptotic lines upon a surface be parametric is that

$$D = D' = 0$$

if these equations hold, and, furthermore, the parametric curves are orthogonal, it is seen from (33) that the mean curvature is zero, conversely. Hence

necessary and sufficient condition that the asymptotic lines form a conjugate system is that the mean curvature of the surface be zero A surface whose mean curvature is zero at every point is called a *minimal surface*. At each of its points the Dupin indicatrix consists of two conjugate equilateral hyperbolas. From (39) we find that the angle between conjugate directions is given by

$$\tan(\theta' - \theta) = \frac{\rho_2 \cot \theta + \rho_1 \tan \theta}{\rho_2 - \rho_1}$$

Consider only real lines, this angle can be zero only for surfaces of negative curvature, in which case the directions are asymptotic. It is natural, therefore, to seek the conjugate directions upon a surface of positive curvature for which the included angle is a minimum. To this end we differentiate the right-hand member of the equation with respect to θ and equate the result to zero. The result is reducible to

$$\tan \theta = \pm \sqrt{\frac{\rho_2}{\rho_1}}$$

From (39) we have

$$\tan \theta' = \mp \sqrt{\frac{\rho_2}{\rho_1}}$$

From these equations it follows that $\theta' = -\theta$, and

$$\tan(\theta' - \theta) = \pm \frac{2\sqrt{\rho_1 \rho_2}}{\rho_2 - \rho_1}$$

Indeed, when $\theta' = -\theta$ equation (39) becomes (41). Hence

on a surface of positive curvature there is a unique conjugate direction at any point for which the angle between the directions is the minimum.

minimum angle between conjugate directions at the point, it is only conjugate system whose directions are symmetric with respect to the directions of the lines of curvature

These lines are called the characteristic lines. It is of interest to note that equations (35) and (41) are similar, and that the asymptotic directions upon a surface of negative curvature are symmetric with respect to the directions of the lines of curvature.

As just seen, if θ is the angle which one characteristic line makes with the line of curvature $v = \text{const.}$ at a point, the other characteristic line makes the angle $-\theta$. Hence the radii of normal curvature for these directions are equal, and consequently a necessary and sufficient condition that the characteristic curves of a surface be parabolic is

$$(42) \quad \frac{D}{E} = \frac{D'}{G}, \quad D' = 0$$

56 Corresponding systems on two surfaces. By reasoning similar to that of § 34 we establish the theorem

A necessary and sufficient condition that the curves defined by $R du^2 + 2S dudv + T dv^2 = 0$ form a conjugate system upon a surface is

$$RD'' + TD - 2SD' = 0$$

From this we have at once

If the second quadratic forms of two surfaces S and S_1 are $D du^2 + 2D' dudv + D'' dv^2$ and $D_1 du^2 + 2D'_1 dudv + D''_1 dv^2$, and a point on one surface is said to correspond to the point on the other with the same values of u and v , the equation

$$(43) \quad \begin{vmatrix} du^2 & D''_1 & D'' \\ -dudv & D'_1 & D' \\ dv^2 & D_1 & D \end{vmatrix} = 0$$

defines a system of curves which is conjugate for both surfaces.

By the methods of § 50 we prove that these curves are real either on both of the surfaces S , S_1 is of positive curvature. If the curvature of S is negative and it is referred to its asymptotic directions the above equation reduces to

$$D_1 du^2 - D''_1 dv^2 = 0$$

ence the system is real when D_1 and D_1'' have the same sign, that is, when the curvature of S_1 is positive

Another consequence of the above theorem is

A necessary and sufficient condition that asymptotic lines on one of two surfaces S, S_1 correspond to a conjugate system on the other is

$$4) \quad DD_1'' + D''D_1 - 2D'D_1' = 0$$

EXAMPLES

1 Find the curves on the general surface of revolution which are conjugate to the loxodromic curves which cut the meridians under the angle α

2 Find the curves on the general right conoid, Ex 1, p 56, which are conjugate to the orthogonal trajectories of the generators

3 When the equations of a surface are of the form

$$x = U_1, \quad y = V_1, \quad z = U_2 + V_2,$$

where U_1 and U_2 are functions of u alone, and V_1 and V_2 of v alone, the parametric curves are plane and form a conjugate system

4 Prove that the sum of normal radii at a point in conjugate directions is constant

5 When a surface of revolution is referred to its meridians and parallels, the asymptotic lines can be found by quadratures

6 Find the asymptotic lines on the surface

$$z = a(1 + \cos u) \cot v, \quad y = a(1 + \cos u), \quad z = \frac{a \cos u}{\sin v}$$

7 Determine the asymptotic lines upon the surface $z = y \sin x$ and their orthogonal trajectories. Show that the z axis belongs to one of the latter families

8 Find the asymptotic lines on the surface $2y^2 - 2xyz + z^2 = 0$ and determine their projections on the xy plane

9 Prove that the product of the normal radii in conjugate directions is a maximum for characteristic lines and a minimum for lines of curvature

10 When the parametric lines are any whatever, the equation of characteristic lines is

$$[D(GD - ED) - 2D'(FD - ED)] du + 2[D(GD + ED) - 2FDD] du dv + [2D(GD - FD) - D'(GD - ED)] dv = 0$$

57 Geodesic curvature Geodesics Consider a curve C upon a surface and the tangent plane to the surface at a point M of C . Project orthogonally upon this tangent plane the portion of the curve in the neighborhood of M , and let C' denote this projection. The curve C' is a normal section of the projecting cylinder, and C is a curve upon the latter, tangent to C' at M . Hence the theorem

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When the second member of equation (55) is developed by (III, 46, 56), we have

$$\begin{aligned} \frac{1}{\rho_s} &= -\frac{1}{H} \left\{ \frac{1}{\sqrt{\Delta_1 \phi}} \left[\frac{\partial}{\partial u} \left(\frac{G \frac{\partial \phi}{\partial u} - F \frac{\partial \phi}{\partial v}}{H} \right) + \frac{\partial}{\partial v} \left(\frac{E \frac{\partial \phi}{\partial v} - F \frac{\partial \phi}{\partial u}}{H} \right) \right] \right. \\ &\quad \left. + \frac{G \frac{\partial \phi}{\partial u} - F \frac{\partial \phi}{\partial v}}{H} \frac{\partial}{\partial u} \frac{1}{\sqrt{\Delta_1 \phi}} + \frac{E \frac{\partial \phi}{\partial v} - F \frac{\partial \phi}{\partial u}}{H} \frac{\partial}{\partial v} \frac{1}{\sqrt{\Delta_1 \phi}} \right\} \\ &= -\frac{1}{H} \left\{ \frac{\partial}{\partial u} \left(\frac{G \frac{\partial \phi}{\partial u} - F \frac{\partial \phi}{\partial v}}{H \sqrt{\Delta_1 \phi}} \right) + \frac{\partial}{\partial v} \left(\frac{E \frac{\partial \phi}{\partial v} - F \frac{\partial \phi}{\partial u}}{H \sqrt{\Delta_1 \phi}} \right) \right\} \end{aligned}$$

Hence we have the formula of Bonnet*

$$\begin{aligned} (56) \quad \frac{1}{\rho_s} &= \frac{1}{H} \left\{ \frac{\partial}{\partial u} \left(\frac{F \frac{\partial \phi}{\partial v} - G \frac{\partial \phi}{\partial u}}{\left[E \left(\frac{\partial \phi}{\partial v} \right)^2 - 2 F \frac{\partial \phi}{\partial v} \frac{\partial \phi}{\partial u} + G \left(\frac{\partial \phi}{\partial u} \right)^2 \right]^{\frac{1}{2}}} \right) \right. \\ &\quad \left. + \frac{\partial}{\partial v} \left(\frac{F \frac{\partial \phi}{\partial u} - E \frac{\partial \phi}{\partial v}}{\left[E \left(\frac{\partial \phi}{\partial v} \right)^2 - 2 F \frac{\partial \phi}{\partial v} \frac{\partial \phi}{\partial u} + G \left(\frac{\partial \phi}{\partial u} \right)^2 \right]^{\frac{1}{2}}} \right) \right\} \end{aligned}$$

In particular, the geodesic curvature of the parametric curves, when the latter do not form an orthogonal system, is given by

$$(57) \quad \begin{cases} \frac{1}{\rho_{su}} = \frac{1}{H} \left(\frac{\partial}{\partial u} \frac{F}{\sqrt{E}} - \frac{\partial}{\partial v} \sqrt{E} \right), \\ \frac{1}{\rho_{sv}} = -\frac{1}{H} \left(\frac{\partial}{\partial v} \frac{F}{\sqrt{G}} - \frac{\partial}{\partial u} \sqrt{G} \right) \end{cases}$$

The geodesic curvature of a curve of the family, defined by the differential equation

has the value

$$(58) \quad \frac{1}{\rho_g} = \frac{1}{H} \left\{ \frac{\partial}{\partial u} \left(\frac{FN - GM}{\sqrt{EN^2 - 2FMN + GM^2}} \right) + \frac{\partial}{\partial v} \left(\frac{FM - EN}{\sqrt{EN^2 - 2FMN + GM^2}} \right) \right\}$$

* Mémoire sur la théorie générale des surfaces, *Journal de l'Ecole Polytechnique* Cahier 32 (1848) p 1

tration of the preceding results, we establish the theorem

The curves of an orthogonal system have constant geodesic curvature, the isothermal

the surface is referred to these lines, and the linear element is written $^2 + Gdv^2$, the condition that the geodesic curvature of these curves be s , by (50) and (51),

$$\frac{1}{\sqrt{EG}} \frac{\partial \sqrt{G}}{\partial u} = U_1, \quad \frac{1}{\sqrt{EG}} \frac{\partial \sqrt{E}}{\partial v} = V_1,$$

and V_1 are functions of u and v respectively. If these equations are differentiated with respect to v and u respectively, we get

$$\frac{\sqrt{G}}{v} - \frac{\partial \log \sqrt{E}}{\partial v} \frac{\partial \log \sqrt{G}}{\partial u} = 0, \quad \frac{\partial^2 \log \sqrt{E}}{\partial u \partial v} - \frac{\partial \log \sqrt{E}}{\partial v} \frac{\partial \log \sqrt{G}}{\partial u} = 0$$

and, we obtain $\frac{\partial^2}{\partial u \partial v} \log \frac{E}{G} = 0$

G is equal to the ratio of a function of u and a function of v , and the isothermal. In terms of isothermal parameters, equations (1) are of

$$\frac{1}{\lambda^2} \frac{\partial \lambda}{\partial u} = U', \quad \frac{1}{\lambda^2} \frac{\partial \lambda}{\partial v} = V',$$

linear element is

$$ds^2 = \frac{du^2 + dv^2}{(U + V)^2}$$

identical that the meridians and parallels on a surface of revolution form an isotherm. The same is true likewise of an orthogonal system of small circles on a sphere.

Geodesic torsion. We have just seen that when a curve is defined by a finite equation or a differential equation, its geodesic curvature can be found directly. The same is true of the normal curvature of the surface in the direction of the curve by (18). From (16) and (47) follow the expressions for ρ and $\bar{\omega}$. In defining the curve it remains for us to obtain an expression for the torsion.

From the definition of $\bar{\omega}$ it follows that

$$\sin \bar{\omega} = X\lambda + Y\mu + Z\nu,$$

where λ, μ, ν are the direction-cosines of the binormal. If this is differentiated with respect to the arc of the curve, the Frenet formulas (I, 50) are used in the reduction, we get

$$\cos \bar{\omega} \left(\frac{d\bar{\omega}}{ds} - \frac{1}{\tau} \right) = \sum \lambda \frac{dX}{ds}$$

From (I, 37, 41) we have

$$\lambda = \rho \left(\frac{dy}{ds} \frac{d^2z}{ds^2} - \frac{dz}{ds} \frac{d^2y}{ds^2} \right), \quad \mu = \rho \left(\frac{dz}{ds} \frac{d^2x}{ds^2} - \frac{dx}{ds} \frac{d^2z}{ds^2} \right),$$

$$\nu = \rho \left(\frac{dx}{ds} \frac{d^2y}{ds^2} - \frac{dy}{ds} \frac{d^2x}{ds^2} \right),$$

and
$$\rho \sum X \frac{d^2x}{ds^2} = \sum X l = \cos \bar{\omega}$$

Moreover, from (13), we obtain the identity

$$\frac{dX}{ds} \frac{dy}{ds} - \frac{dY}{ds} \frac{dx}{ds} = \frac{Z}{H} \left[(ED' - FD) \left(\frac{du}{ds} \right)^2 + (ED'' - GD) \frac{du}{ds} \frac{dv}{ds} + (FD'' - GD') \left(\frac{dv}{ds} \right)^2 \right]$$

Consequently equation (60) is equivalent to

$$(61) \quad \cos \bar{\omega} \left(\frac{1}{\tau} - \frac{d\bar{\omega}}{ds} - \frac{1}{T} \right) = 0,$$

where $1/T$ has the value

$$(62) \quad \frac{1}{T} = \frac{(FD - ED') du^2 + (GD - ED'') du dv + (GD' - FD'') dv^2}{H(Edu' + 2F du dv + G dv^2)}$$

When $\cos \bar{\omega}$ is different from zero, that is, when the curve is not an asymptotic line, equation (61) becomes

$$(63) \quad \frac{1}{\tau} - \frac{d\bar{\omega}}{ds} = \frac{1}{T}$$

As the expression for T involves only the fundamental coefficients and dv/du , we have the following theorem of Bonnet

The function $\frac{1}{\tau} - \frac{d\bar{\omega}}{ds}$ is the same for all curves which have the same tangent at a common point

Among these curves there is one geodesic, and only one, for it will be shown later (§ 85) that one geodesic and only one passes through a given point and has a given direction at the point. At every point of this geodesic $\bar{\omega}$ is equal to 0° or 180° , and consequently $\tau = T$. Hence the value of T for a given point and direction is that of the radius of torsion of the geodesic with this direction. The function T is therefore called the *radius of geodesic torsion* of

urve From (63) it is seen that T is the radius of torsion of urve whose osculating plane makes a constant angle with the nt plane *

hen the numerator of the right-hand member of equation (62) uated to zero, we have the differential equation of lines of ture Hence

necessary and sufficient condition that the geodesic torsion of ve be zero at a point is that the curve be tangent to a line of ture at the point

ie geodesic torsion of the parametric lines is given by

$$\frac{1}{T_u} = \frac{FD - ED'}{EH}, \quad \frac{1}{T_v} = \frac{GD' - FD''}{GH}$$

n these lines form an orthogonal system T_u and T_v differ only gn Consequently the geodesic torsion at the point of meeting o curves cutting orthogonally is the same to within the sign us far in the consideration of equation (61) we have excluded ase of asymptotic lines In considering them now, we assume they are parametric The direction-cosines of the tangent and mal to a curve $v = \text{const}$ in this case are

$$\alpha = \frac{1}{\sqrt{E}} \frac{\partial x}{\partial u}, \quad \beta = \frac{1}{\sqrt{E}} \frac{\partial y}{\partial u}, \quad \gamma = \frac{1}{\sqrt{E}} \frac{\partial z}{\partial u},$$

$$\lambda = \epsilon X, \quad \mu = \epsilon Y, \quad \nu = \epsilon Z,$$

e ϵ is $+1$ or -1 Consequently the direction-cosines of the ipal normal have the values

$$l = \frac{\epsilon}{\sqrt{E}} \left(Y \frac{\partial z}{\partial u} - Z \frac{\partial y}{\partial u} \right),$$

similar expressions for m and n

hen in the Frenet formulas

$$\frac{d\lambda}{ds} = \frac{l}{\tau}, \quad \frac{d\mu}{ds} = \frac{m}{\tau}, \quad \frac{d\nu}{ds} = \frac{n}{\tau},$$

ubstitute the above values, and in the reduction make use 1) and (13), we get

$$\frac{1}{\tau} = -\frac{D'}{H} = -\sqrt{-K}$$

hus far exception must be made of asymptotic lines but later this restriction will oved

In like manner, the torsion of the asymptotic lines $u = \text{const}$ is found to be $\sqrt{-K}$. But from (64) we find that the geodesic torsion in the direction of the asymptotic lines is $\mp \sqrt{-K}$. Hence equation (63) is true for the asymptotic lines as well as for all other curves on the surface.

Incidentally we have established the following theorem of Enneper

The square of the torsion of a real asymptotic line at a point is equal to the absolute value of the total curvature of the surface at the point, the radii of torsion of the asymptotic lines through a point differ only in sign

The following theorem of Joachimsthal is an immediate consequence of (63)

When two surfaces meet under a constant angle, the line of intersection is a line of curvature of both or neither, and conversely, when the curve of intersection of two surfaces is a line of curvature of both they meet under constant angle

For, if $\bar{\omega}_1, \bar{\omega}_2$ denote the values of $\bar{\omega}$ for the two surfaces, and T_1, T_2 the values of T , we have, by subtracting the two equations of the form (63), that $T_1 = T_2$, which proves the first part of the theorem. Conversely, if $1/T_1 = 1/T_2 = 0$, we have $\frac{d}{ds}(\omega_1 - \omega_2) = 0$, and consequently the surfaces meet under constant angle

EXAMPLES

1 Show that the radius of geodesic curvature of a parallel on a surface of revolution is the same at all points of the parallel, and determine its geometrical significance

2 Find the geodesic curvature of the parametric lines on the surface

$$x = \frac{a}{2}(u+v), \quad y = \frac{b}{2}(u-v), \quad z = \frac{uv}{2}$$

3 Given a family of loxodromic curves upon a surface of revolution which cut the meridians under the same angle α , show that the geodesic curvature of all these curves is the same at their points of intersection with a parallel

4 Straight lines on a surface are the only asymptotic lines which are geodesics

5 Show that the geodesic torsion of a curve is given by

$$\frac{1}{T} = \frac{1}{2} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \sin 2\theta,$$

where θ denotes the angle which the direction of the curve at a point makes with the line of curvature $v = \text{const}$ through the point

6 Every geodesic line of curvature is a plane curve

7 Every plane geodesic line is a line of curvature

8 When a surface is cut by a plane or a sphere under constant angle, it is a line of curvature on the surface, and conversely

9 If the curves of one family of an isothermal orthogonal system have constant geodesic curvature, the curves of the other family have the same property

spherical representation In the discussion of certain properties of a surface S it is of advantage to make a representation of the unit sphere * by drawing radii of the sphere parallel to the directions of the normals to S , and taking the extremities of the radii as *spherical images* of the corresponding points on S . If a point M moves along a curve on S , its image m describes a curve on the sphere. If we limit our consideration to a portion of the surface in which no two normals are parallel, the portion of the surface and sphere will be in a one-to-one correspondence. The mapping of the surface upon the sphere is called the *spherical representation* of the surface, or the *Gaussian representation*. It is employed by Gauss in his treatment of the curvature of

†
The coordinates of m are the direction-cosines of the normal to the surface, namely X, Y, Z , so that if we put

$$\mathcal{E} = \sum \left(\frac{\partial X}{\partial u} \right)^2, \quad \mathcal{F} = \sum \frac{\partial X}{\partial u} \frac{\partial X}{\partial v}, \quad \mathcal{G} = \sum \left(\frac{\partial X}{\partial v} \right)^2,$$

the area of the linear element of the spherical representation is

$$d\sigma^2 = \mathcal{E} du^2 + 2\mathcal{F} du dv + \mathcal{G} dv^2$$

As we established the following equations

$$\begin{cases} \frac{\partial X}{\partial u} = \frac{FD' - GD}{H^2} \frac{\partial x}{\partial u} + \frac{FD - ED'}{H^2} \frac{\partial x}{\partial v}, \\ \frac{\partial X}{\partial v} = \frac{FD'' - GD'}{H^2} \frac{\partial x}{\partial u} + \frac{FD' - ED''}{H^2} \frac{\partial x}{\partial v} \end{cases}$$

and similar ones in Y and Z , the functions $\mathcal{E}, \mathcal{F}, \mathcal{G}$ may be given the forms

$$\begin{cases} \mathcal{E} = \frac{1}{H^2} [GD^2 - 2FDD' + ED'^2], \\ \mathcal{F} = \frac{1}{H^2} [GDD' - F(DD'' + D''D) + ED'D''], \\ \mathcal{G} = \frac{1}{H^2} [GD'^2 - 2FD'D'' + ED''^2], \end{cases}$$

expressions of the total and mean curvatures (§ 52),

$$\mathcal{E} = K_m D - KE, \quad \mathcal{F} = K_m D' - KF, \quad \mathcal{G} = K_m D'' - KG$$

sphere of unit radius and center at the origin of coordinates † I c p 9

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In consequence of these relations the linear element (67) may be given the form

$$(71) \quad d\sigma^2 = K_m(D du^2 + 2D' dudv + D'' dv^2) - K(E du^2 + 2F dudv + G dv^2),$$

and, by (18),

$$(72) \quad d\sigma^2 = \left(\frac{K_m}{R} - K \right) ds^2$$

From (70) we have also

$$(73) \quad H = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2} = \epsilon KH,$$

where ϵ is ± 1 , according as K is positive or negative

Equations (69) are linear in E, F, G . Solving for the latter, we have

$$(74) \quad \begin{cases} E = \frac{1}{H^2} [\mathcal{E}D^2 - 2\mathcal{F}DD' + \mathcal{E}D'^2], \\ F = \frac{1}{H^2} [\mathcal{E}DD' - \mathcal{F}(DD'' + D'^2) + \mathcal{E}D'D''], \\ G = \frac{1}{H^2} [\mathcal{E}D''^2 - 2\mathcal{F}D'D'' + \mathcal{E}D'''^2] \end{cases}$$

In seeking the differential equation of the lines of curvature from the definition that the normals to the surface along such a curve form a developable surface, we found (§ 51) that for a displacement in the direction of a line of curvature we have

$$\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + r \left(\frac{\partial X}{\partial u} du + \frac{\partial X}{\partial v} dv \right) = 0,$$

and similar equations in y and z , where r denotes the radius of principal curvature for the direction. If these equations be multiplied respectively by $\frac{\partial X}{\partial u}, \frac{\partial Y}{\partial u}, \frac{\partial Z}{\partial u}$ and added, and likewise by $\frac{\partial X}{\partial v}, \frac{\partial Y}{\partial v}, \frac{\partial Z}{\partial v}$ and added, the resulting equations may be written

$$D du + D' dv - r(\mathcal{E} du + \mathcal{F} dv) = 0,$$

$$D' du + D'' dv - r(\mathcal{F} du + \mathcal{G} dv) = 0$$

Eliminating r , we have as the equation of the lines of curvature

$$(75) \quad (D\mathcal{F} - D'\mathcal{E}) du^2 + (D\mathcal{G} - D''\mathcal{E}) dudv + (D'\mathcal{G} - D'''\mathcal{F}) dv^2 = 0$$

the elimination of du and dv gives the equation of the 1 radiu in the form

$$5\mathcal{E} - \mathcal{F}^2)r^2 - (\mathcal{E}D'' + \mathcal{E}D - 2\mathcal{F}D')r + (DD'' - D'^2) = 0,$$

$$\begin{cases} \rho_1 + \rho_2 = \frac{\mathcal{E}D'' + \mathcal{E}D - 2\mathcal{F}D'}{f^2}, \\ \rho_1\rho_2 = \frac{DD'' - D'^2}{f^2} \end{cases}$$

esults enable us to write equations (74) thus

$$\begin{cases} E = (\rho_1 + \rho_2)D - \rho_1\rho_2\mathcal{E}, \\ F = (\rho_1 + \rho_2)D' - \rho_1\rho_2\mathcal{F}, \\ G = (\rho_1 + \rho_2)D'' - \rho_1\rho_2\mathcal{G} \end{cases}$$

elations between a surface and its spherical representation
the radius of normal curvature R is a function of the direc-
ept when the surface is a sphere, we obtain from (72) the
g theorem

*ecessary and sufficient condition that the spherical representation
face be conformal is that the surface be minimal or a sphere*

consequence of this theorem we have that every orthog-
stem on a minimal surface is represented on the sphere by
ogonal system From (70) it is seen that if a surface is
imal, the parametric systems on both the surface and the
an be orthogonal only when D' is zero, that is, when the
curvature are parametric Hence we have

*ines of curvature of a surface are represented on the sphere
rthogonal system, this is a characteristic property of lines
ture, unless the surface be minimal*

heorem follows also as a direct consequence of the theorem

*ecessary and sufficient condition that the tangents to a curve
urface and to its image at corresponding points be parallel is
curve be a line of curvature*

der to prove this theorem we assume that the curve is
ric, $v = \text{const}$ Then the condition of parallelism is

$$\frac{\partial X}{\partial u} \frac{\partial Y}{\partial u} \frac{\partial Z}{\partial u} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} \frac{\partial z}{\partial u}$$

From (68) it follows that in this case $(FD - ED')$ must be zero. But the latter is the condition that the curves $v = \text{const}$ be lines of curvature (§ 51). Moreover, from (32) it follows that the positive half-tangents to a line of curvature and its spherical representation have the same or contrary sense according as the corresponding radius of normal curvature is negative or positive.

In consequence of (7) the equation (40) of the asymptotic directions may be written

$$dx dX + dy dY + dz dZ = 0$$

And, so we have the theorem

The tangents to an asymptotic line and to its spherical representation at corresponding points are perpendicular to one another, this property is characteristic of asymptotic lines

It is evident that the direction-cosines of the normal to the sphere are equal to X, Y, Z , to within sign at most. Let them be denoted by $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, then

$$(79) \quad \mathcal{X} = \frac{1}{\mathcal{R}} \left(\frac{\partial Y}{\partial u} \frac{\partial Z}{\partial v} - \frac{\partial Z}{\partial u} \frac{\partial Y}{\partial v} \right)$$

When expressions similar to (68) are substituted for the quantities in the parentheses, the latter expression is reducible to $AA'A$. Hence, in consequence of (73), we have

$$(80) \quad \mathcal{X} = \epsilon X, \quad \mathcal{Y} = \epsilon Y, \quad \mathcal{Z} = \epsilon Z,$$

where $\epsilon = \pm 1$ according as the curvature of the surface is positive or negative.

From the above it follows that according as a point of a surface is elliptic or hyperbolic the positive sides of the tangent planes at corresponding points of the surface and the sphere are the same or different. Suppose, for the moment, that the lines of curvature are parametric. From our convention about the positive direction of the normal to a surface, and the above results, it follows that both the tangents to the parametric curves through a point M have the same sense as the corresponding tangents to the sphere, or both have the opposite sense, when M is an elliptic point; but that one tangent has the same sense as the corresponding tangent to the sphere, and the other the opposite sense, when the point is

Hence, when a point describes a closed curve on a surface its image describes a closed curve on the sphere in the same or opposite sense according as the surface has positive or negative curvature. We say that the areas inclosed by these curves have the same or opposite signs in these respective cases. Suppose now that we consider a small parallelogram on the surface whose vertices are the points (u, v) , $(u + du, v)$, $(u, v + dv)$, $(u + du, v + dv)$. The vertices of the corresponding parallelogram on the sphere have the same curvilinear coordinates, and the areas are $H du dv$ and $\epsilon / H du dv$, where $\epsilon = \pm 1$ according as the surface has positive or negative curvature in the neighborhood of the point (u, v) . The limiting value of the ratio of the spherical and surface areas as the vertices of the latter approach the point (u, v) is a measure of the curvature of the surface similar to that of a plane curve. In consequence of (73) this limiting value is the Gaussian curvature K . Since any closed area may be looked upon as made up of such small parallelograms, we have the following theorem of Gauss

The limit of the ratio of the area of a closed portion of a surface to the area of the spherical image of it, as the former converges to a point, is equal in value to the product of the principal radii at the point.

Since the normals to a developable surface along a generator are parallel, there can be no closed area for which there are not two normals which are parallel. Hence spherical representation, as defined in § 60, applies only to nondevelopable surfaces, but so far as the Gauss-Bonnet theorem goes, it is not necessary to make this exception, the total curvature of a developable surface is zero (§ 64), and the area of the spherical image of any closed area on such a surface is zero.

The fact that the Gaussian curvature is zero at all points of a developable surface, and such a surface is surely curved, makes this measure not altogether satisfactory, and so others have been suggested. Thus, Sophie Germain* advocated mean curvature, and Casorati† has put forward the expression $\frac{1}{2} \left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} \right)$. According to the first, the curvature of a minimal surface is zero, and according to the second, a minimal surface has the same curvature as a sphere. Hence the Gaussian curvature continues to be the one most frequently used, which may be largely due to an important property of it to be discussed later (§ 64).

62 Helicoids We apply the preceding results in a study of an important class of surfaces called the *helicoids*. A helicoid is generated by a curve, plane or twisted, which is rotated about a fixed line as axis, and at the same time translated in the direction of the axis with a velocity which is in constant ratio with the velocity of rotation. A section of the surface by a plane through the axis is called a *meridian*. All the meridians are equal plane curves, and the surface can be generated by a meridian moving with the same velocities as the given curve. The particular motion described is called helicoidal motion, and so we may say that any helicoid can be generated by a plane curve with helicoidal motion.

In order to determine the equations of a helicoid in parametric form, we take the axis of rotation for the z -axis, and let u denote the distance of a point of the surface from the axis, and v the angle made by the plane through the point and the axis with the xz -plane in the positive direction of rotation. If the equation of the generating curve in any position of its plane is $z = \phi(u)$, the equations of the surface are

$$(81) \quad x = u \cos v, \quad y = u \sin v, \quad z = \phi(u) + av,$$

where a denotes the constant ratio of the velocities, it is called the *parameter* of the helicoidal motion. When, in particular, a is zero, these equations define any surface of revolution. Moreover, when $\phi(u)$ is a constant, the curves $v = \text{const}$ are straight lines perpendicular to the axis and so the surface is a right conoid. It is called the *right helicoid*.

By calculation we obtain from (81)

$$(82) \quad I = 1 + \phi', \quad I - a\phi', \quad G = u^2 + a^2,$$

where the accent indicates differentiation with respect to u . From the method of generation it follows that the curves $v = \text{const}$ are meridians and $u = \text{const}$ are helices on the helicoids and circles on surfaces of revolution. From (82) it is seen that these curves form an orthogonal system only on surfaces of revolution and on the right helicoid. Moreover from (57) it is found that the geodesic curvature of the meridians is zero only when a is zero or ϕ' is a constant. In the latter case the meridian is a straight line perpendicular to the axis or oblique, according as ϕ' is zero or not.

the meridians of surfaces of revolution and of the ruled surfaces are geodesics

orthogonal trajectories of the helices upon a helicoid are defined by the equation (cf III, 31)

$$a\phi'du + (u^2 + a^2)dv = 0$$

if we put $v_1 = \int \frac{a\phi'}{u^2 + a^2} du + v$,

curves $v_1 = \text{const}$ are the orthogonal trajectories, and their equations in finite form are found by a quadrature. In terms of parameters u and v_1 the linear element is

$$ds^2 = \left(1 + \frac{u^2\phi'^2}{u^2 + a^2}\right) du^2 + (u^2 + a^2) dv_1^2$$

As an immediate consequence of this result we have that the meridians and their orthogonal trajectories on any helicoid form an orthogonal system

(83) and (§ 46) we have the theorem of Bour

A helicoid is applicable to some surface of revolution, and the meridians correspond to the parallels on the latter

We derive also the following expressions

$$\lambda, \mu, Z = \frac{a \sin v - u\phi' \cos v, -(a \cos v + u\phi' \sin v), u}{\sqrt{u^2(1 + \phi'^2) + a^2}},$$

$$D, D', D'' = \frac{u\phi'', -a, u\phi'}{\sqrt{u^2(1 + \phi'^2) + a^2}}$$

(84) it follows that a meridian is a normal section of a surface of revolution at all its points, and consequently is a line of curvature (Ex 7, p. 140). This is evident also from the equations of curvature of a helicoid, namely

$$1 + \phi' + u\phi'\phi'' \mid du + [(u + a)u\phi'' - (1 + \phi')u\phi'] du dv - a[u\phi'' + u + a] dv^2 = 0$$

and, the meridians are lines of curvature of those helicoids, for which ϕ satisfies the condition

$$1 + \phi'^2 + u\phi'\phi'' = 0$$

By integration this gives

$$\phi = \sqrt{c^2 - u^2} - c \log \frac{c + \sqrt{c^2 - u^2}}{u}$$

When the surface is the right helicoid the expressions for D and D'' vanish. Hence the meridians and helices are the asymptotic lines. Moreover, these lines form an orthogonal system, so that the surface is a minimal surface (§ 55). Since the tangent planes to a surface along an asymptotic line are its osculating planes, if the surface is a ruled minimal surface, the generators are the principal normals of all the curved asymptotic lines.

But a circular helix is the only Bertrand curve whose principal normals are the principal normals of an infinity of curves (§ 19). Hence we have the theorem of Catalan

The right helicoid is the only real minimal ruled surface.

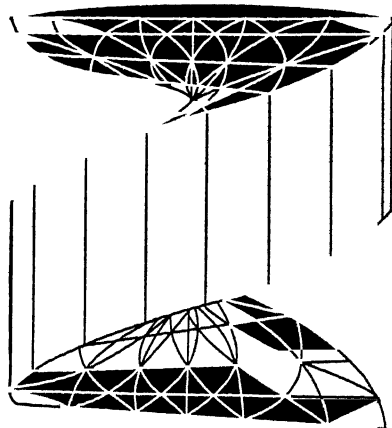


FIG. 16

In fig. 16 are represented the asymptotic lines and lines of curvature of a right helicoid.

For any other helicoid the equation of the asymptotic lines is

$$(87) \quad u\phi'' du^2 - 2u\phi' du dv + u^2\phi' dv^2 = 0$$

As the coefficients in (86) and (87) are functions of u alone, we have the theorem

When a helicoid is referred to its meridians and helices the asymptotic lines and the lines of curvature can be found by quadratures.

EXAMPLES

1. Show that the spherical representation of the lines of curvature of a surface of revolution is isothermal.
2. The osculating planes of a line of curvature and of its spherical representation at corresponding points are parallel.
3. The angles between the asymptotic directions at a point on a surface and between their spherical representation are equal or supplementary, according as the surface has positive or negative curvature at the point.

that the helicoidal surface

$$x = u \cos v, \quad y = u \sin v, \quad z = bv + \int \left[\frac{(u^2 + b^2)}{(u^2 - b^2)} \right]^{\frac{1}{2}} \frac{du}{u}$$

total curvature of a helicoid is constant along a helix

orthogonal trajectories of the helices upon a helicoid are geodesics

fundamental functions E, F, G of a surface are functions of a single u , the surface is applicable to a surface of revolution

the equations of the helicoid generated by a circle of constant radius b which passes through the axis and the lines of curvature on the surface, also the equations of the surface in terms of parameters referring to the meridians and orthogonal trajectories

GENERAL EXAMPLES

a pencil of planes be drawn through a tangent MT to a surface, and if the normals at M to the sections of the surface by these planes are equal to the curvature of the sections, the locus of the end points is a line normal to the plane determined by MT and the normal to the surface at M

P is a point of a developable surface, P_0 the point where the generator through P touches the edge of regression, t the length P_0P , ρ and τ the radius of curvature and torsion of the edge of regression, then the principal radii of curvature are given by

$$\frac{1}{\rho_1} = 0, \quad \frac{1}{\rho_2} = \frac{\rho}{\tau t}$$

the surface of revolution of a parabola about its directrix, the principal radii of curvature in constant ratio

the equations $x = a \cos u$, $y = a \sin u$, $z = uv$ define a family of circular helices which pass through the point $A(a, 0, 0)$ of the cylinder, each helix has an involute whose points are at the distance c from A (cf I, 106). Find the surface which is the locus of these involutes, show that the tangents to the helices are principal directions on this surface, find also the lines of curvature upon the latter

a surface defined by the equations (cf § 25)

$$1 + p^2 + q^2 = q^2 f(y), \quad x + pz = \phi(p)$$

system of lines of curvature in planes parallel to the xz plane and to the yz plane respectively

the equations

$$y - \alpha x = 0, \quad x^2 + y^2 + z^2 - 2\beta x - \alpha^2 = 0,$$

and β are parameters, define all the circles through the points $(0, 0, \alpha)$, α being a constant. Show that the circles determined by a relation $\beta = f(\alpha)$ are the principal circles of a family of spheres, except when $f(\alpha)$ is a linear function, in which case the circles are lines of curvature on the envelope of these spheres

one of the lines of curvature of a developable surface lies upon a sphere, the nonrectilinear lines of curvature lie on concentric spheres

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8 If P is a point on a surface, P_0 the center of normal curvature of the line bisecting the angle between the lines of curvature, and P_1, P_2 the centers of normal curvature in two directions equally inclined to the first, then the four points P, P_1, P_0, P_2 form a harmonic range.

9 If $R_1, R_2, R_3, \dots, R_m$ denote the radii of normal curvature of m sections of a surface which make equal angles $2\pi/m$ with one another, and $m > 2$, then

$$\frac{1}{m} \left(\frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_m} \right) = \frac{1}{2} \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right)$$

10 If the Dupin indicatrix at a point P of a surface is an ellipse, and through either one of the asymptotes of its focal hyperbola two planes be drawn perpendicular to one another, their intersections with the tangent plane are conjugate directions on the surface

11 All curves tangent to an asymptotic line at a point M , and whose osculating planes are not tangent to the surface at M , have M for a point of inflection.

12 The normal curvature of an orthogonal trajectory of an asymptotic line is equal to the mean curvature of the surface at the point of intersection

13 The surface of revolution whose equations are

$$x = u \cos v, \quad y = u \sin v, \quad z = a \log(u + \sqrt{u^2 - a^2})$$

is generated by the rotation of a catenary about its axis. It is called the *catenoid*. Show that it is the only minimal surface of revolution.

14 When the osculating plane of a line of curvature makes a constant angle with the tangent plane to the surface, the line of curvature is plane.

15 A plane line of curvature is represented on the unit sphere by a circle.

16 The cylinder whose right section is the curve defined by the intrinsic equation $\rho = a - s^2/b$ where a and b are positive constants, has the characteristic property that upon it lie curves of curvature $\frac{1}{\sqrt{ab}}$ whose geodesic curvature is $1/\sqrt{ab}$.

17 When a surface is referred to an orthogonal system of lines, and the radii of geodesic curvature of the curves $u = \text{const}$ and $v = \text{const}$ are ρ_u and ρ_v respectively, the geodesic curvature of the curve which makes an angle θ with the lines $v = \text{const}$ is given by

$$\frac{1}{\rho} = \frac{1}{\rho_u} \cos^2 \theta + \frac{1}{\rho_v} \sin^2 \theta$$

18 When a surface is referred to an orthogonal system of lines, and ρ_u and ρ_v denote the radii of geodesic curvature, and the u and v are the systems of orthogonal trajectories of the parameter lines, and f_u and f_v the scalar functions for the orthogonal trajectories of the former, then whatever be the direction of the first curves the quantity $\frac{d}{ds} \left(\frac{1}{\rho} \right) = \frac{1}{\rho^2} \left(\frac{df_u}{ds} \rho_u + \frac{df_v}{ds} \rho_v \right)$ is constant at a point.

19 If ρ and ρ' denote the radii of curvature of a line of curvature and its spherical representation, and also ρ_u and ρ_v the radii of curvature of these curves, then

$$\frac{ds}{\rho} = \frac{ds'}{\rho'} = \frac{ds_u}{\rho_u} = \frac{ds_v}{\rho_v}$$

where ds and ds' are the line elements of the curves.

When a surface is referred to its lines of curvature, and θ_0, θ'_0 denote the such a curve on the surface and its spherical representation make with $v = \text{const}$, the radii of geodesic curvature of these curves, denoted by ρ_0, ρ'_0 respectively, are in the relation

$$d\theta_0 - \frac{ds}{\rho_0} = d\theta'_0 - \frac{ds}{\rho'_0}$$

When the curve

$$x = f(u) \cos u, \quad y = f(u) \sin u, \quad z = -\frac{1}{a} \int f^2(u) du$$

is subjected to a helicoidal motion of parameter a about the z -axis, the various positions of this curve are orthogonal trajectories of the helices, and also geodesics of the surface.

When a curve is subjected to a continuous rotation about an axis, and at the same time to a homothetic transformation with respect to a point of the axis, the tangent to the locus described by a point of the curve makes a constant angle with the axis, the locus of the resulting curves is called a *spiral surface*. If the z -axis be taken for the axis of rotation and the origin for the center of homothetic transformation, the equations of the surface are of the form

$$x = f(u) e^{hu} \cos(u + v), \quad y = f(u) e^{hu} \sin(u + v), \quad z = \phi(u) e^{kv},$$

where h, k are constants.

A spiral surface can be generated in the following manner. Let C be a curve in any line, and P a point on the latter, if each point M on C describes an arc of a circle with vertex P and axis l such that the perpendicular upon l , from the moving point M , revolves about l with constant velocity, the locus of these curves is a spiral surface (cf. Ex. 5, § 38).

Show that the orthogonal trajectories of the curves $u = \text{const}$, in Ex. 22, can be found by quadratures, and that the linear element can be put in the form

$$ds^2 = e^{2\beta} (d\alpha^2 + A^2 d\beta^2),$$

where β is a function of α alone.

Show that the lines of curvature, minimal lines, and asymptotic lines upon the surface can be found by quadrature.

CHAPTER V

FUNDAMENTAL EQUATIONS THE MOVING TRIHEDRAL

63 Christoffel symbols In this chapter we derive the necessary and sufficient equations of condition to be satisfied by six functions, E, F, G, D, D', D'' , in order that they may be the fundamental quantities for a surface

For the sake of brevity we make use of two sets of symbols, suggested by Christoffel,* which represent certain functions of the coefficients of a quadratic differential form and their derivatives of the first order. If the differential form is

$$a_{11}du_1^2 + 2a_{12}du_1du_2 + a_{22}du_2^2,$$

the first set of symbols is defined by

$$\left[\begin{matrix} ik \\ l \end{matrix} \right] = \frac{1}{2} \left(\frac{\partial a_{il}}{\partial u_k} + \frac{\partial a_{kl}}{\partial u_i} - \frac{\partial a_{ik}}{\partial u_l} \right),$$

where each of the subscripts i, k, l has one of the values 1 and 2 †. From this definition it follows that

$$\left[\begin{matrix} ik \\ l \end{matrix} \right] = \left[\begin{matrix} kl \\ l \end{matrix} \right]$$

When these symbols are used in connection with the first fundamental quadratic form of a surface $ds^2 = E du^2 + 2F du dv + G dv^2$, they are found to have the following significance

$$(1) \quad \left\{ \begin{array}{ll} \left[\begin{matrix} 11 \\ 1 \end{matrix} \right] = \frac{1}{2} \frac{E}{E}, & \left[\begin{matrix} 11 \\ 2 \end{matrix} \right] = \frac{1}{2} \frac{F}{E} - \frac{1}{2} \frac{F}{G}, \\ \left[\begin{matrix} 12 \\ 1 \end{matrix} \right] = \frac{1}{2} \frac{F}{E}, & \left[\begin{matrix} 12 \\ 2 \end{matrix} \right] = \frac{1}{2} \frac{F}{G} - \frac{1}{2} \frac{G}{G}, \\ \left[\begin{matrix} 22 \\ 1 \end{matrix} \right] = \frac{1}{2} \frac{F}{E} - \frac{1}{2} \frac{G}{G}, & \left[\begin{matrix} 22 \\ 2 \end{matrix} \right] = \frac{1}{2} \frac{G}{G} - \frac{1}{2} \frac{G}{G} \end{array} \right.$$

* *Crelle* Vol. LXX pp. 241-245

† This equation defines these symbols for a quadratic form of any number of variables $u_1, \dots, u_n$. In this case i, k, l take the values 1, $\dots$, n .

second set of symbols is defined by the equation

$$\left\{ \begin{smallmatrix} ik \\ \nu \end{smallmatrix} \right\} = A_{\nu 1} \left[\begin{smallmatrix} ik \\ 1 \end{smallmatrix} \right] + A_{\nu 2} \left[\begin{smallmatrix} ik \\ 2 \end{smallmatrix} \right],$$

$\left\{ \begin{smallmatrix} ik \\ \nu \end{smallmatrix} \right\}$ denotes the algebraic complement of $a_{\nu i}$ in the discriminant $a_{22} - a_{12}^2$ divided by the discriminant itself. With reference to the first fundamental quadratic form these symbols mean

$$\begin{aligned} A_{11} &= \frac{G}{H^2}, & A_{12} &= \frac{-F}{H^2}, & A_{22} &= \frac{E}{H^2}, \\ \left\{ \begin{smallmatrix} 11 \\ 1 \end{smallmatrix} \right\} &= \frac{G \frac{\partial E}{\partial u} + F \frac{\partial E}{\partial v} - 2F \frac{\partial F}{\partial u}}{2H^2}, & \left\{ \begin{smallmatrix} 11 \\ 2 \end{smallmatrix} \right\} &= \frac{-F \frac{\partial E}{\partial u} - E \frac{\partial E}{\partial v} + 2E \frac{\partial F}{\partial u}}{2H^2}, \\ \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\} &= \frac{G \frac{\partial E}{\partial v} - F \frac{\partial G}{\partial u}}{2H^2}, & \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\} &= \frac{E \frac{\partial G}{\partial u} - F \frac{\partial E}{\partial v}}{2H^2}, \\ \left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\} &= \frac{-F \frac{\partial G}{\partial v} - G \frac{\partial G}{\partial u} + 2G \frac{\partial F}{\partial v}}{2H^2}, & \left\{ \begin{smallmatrix} 22 \\ 2 \end{smallmatrix} \right\} &= \frac{E \frac{\partial G}{\partial v} + F \frac{\partial G}{\partial u} - 2F \frac{\partial F}{\partial v}}{2H^2} \end{aligned}$$

from these equations we derive the following identities

$$\frac{\partial \log H}{\partial u} = \left\{ \begin{smallmatrix} 11 \\ 1 \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}, \quad \frac{\partial \log H}{\partial v} = \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} 22 \\ 2 \end{smallmatrix} \right\}$$

From the aid of these identities we derive from (III, 15, 16) the following relations

$$-\frac{1}{H} \left(\frac{1}{F} \left\{ \begin{smallmatrix} 11 \\ 2 \end{smallmatrix} \right\} + \frac{1}{G} \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\} \right), \quad \frac{\partial \omega}{\partial v} = -H \left(\frac{1}{E} \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\} + \frac{1}{G} \left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\} \right)$$

From the above definition of the symbols $\left\{ \begin{smallmatrix} ik \\ \nu \end{smallmatrix} \right\}$ we obtain the following important relation

$$\left[\begin{smallmatrix} ik \\ l \end{smallmatrix} \right] = a_{l1} \left\{ \begin{smallmatrix} ik \\ 1 \end{smallmatrix} \right\} + a_{l2} \left\{ \begin{smallmatrix} ik \\ 2 \end{smallmatrix} \right\}$$

The equations of Gauss and of Codazzi. The first two of these (IV, 10) and the equation

$$X \frac{\partial^2 x}{\partial u^2} + Y \frac{\partial^2 y}{\partial u^2} + Z \frac{\partial^2 z}{\partial u^2} = D$$

form a consistent set of equations linear in $\frac{\partial^2 x}{\partial u^2}$, $\frac{\partial^2 y}{\partial u^2}$, $\frac{\partial^2 z}{\partial u^2}$, and the determinant is equal to H . Solving for $\frac{\partial^2 x}{\partial u^2}$, we get

$$\frac{\partial^2 x}{\partial u^2} = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} \frac{\partial x}{\partial u} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \frac{\partial x}{\partial v} + DX,$$

similar equations hold for y and z . Proceeding in like manner with the other equations (IV, 10) and

$$(6) \quad \sum X \frac{\partial^2 x}{\partial u \partial v} = D', \quad \sum X \frac{\partial^2 x}{\partial v^2} = D'',$$

we get the following equations of Gauss

$$(7) \quad \begin{cases} \frac{\partial^2 x}{\partial u^2} = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} \frac{\partial x}{\partial u} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \frac{\partial x}{\partial v} + DX, \\ \frac{\partial^2 x}{\partial u \partial v} = \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \frac{\partial x}{\partial u} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \frac{\partial x}{\partial v} + D'X, \\ \frac{\partial^2 x}{\partial v^2} = \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} \frac{\partial x}{\partial u} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} \frac{\partial x}{\partial v} + D''X \end{cases}$$

For convenience of reference we recall from § 48 the equations

$$(8) \quad \begin{cases} \frac{\partial X}{\partial u} = \frac{FD' - GD}{H^2} \frac{\partial x}{\partial u} + \frac{FD - ED'}{H} \frac{\partial x}{\partial v}, \\ \frac{\partial X}{\partial v} = \frac{FD'' - GD'}{H^2} \frac{\partial x}{\partial u} + \frac{FD' - ED''}{H} \frac{\partial x}{\partial v} \end{cases}$$

The conditions of integrability of the Gauss equations (7) are

$$\frac{\partial}{\partial v} \left(\frac{\partial^2 x}{\partial u^2} \right) = \frac{\partial}{\partial u} \left(\frac{\partial^2 x}{\partial u \partial v} \right), \quad \frac{\partial}{\partial v} \left(\frac{\partial^2 x}{\partial u \partial v} \right) = \frac{\partial}{\partial u} \left(\frac{\partial^2 x}{\partial v^2} \right)$$

By means of (7) and (8) these equations are reducible to the forms

$$(9) \quad a_1 \frac{\partial r}{\partial u} + b_1 \frac{\partial r}{\partial v} + c_1 \lambda = 0, \quad a \frac{\partial r}{\partial u} + b_2 \frac{\partial r}{\partial v} + c \lambda = 0,$$

where $a_1, a_2, \dots, c_2$ are determinate functions of u, v, G, D, D', D'' and their derivatives. Since equations similar to (9) hold for y and z , we must have

$$(10) \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = 0, \quad c_1 = 0, \quad c_2 = 0$$

he expressions for a_1 , a_2 , b_1 , and b_2 are calculated, it is at the first four equations are equivalent to the following

$$\frac{D'' - D'^2}{H^2} F = \frac{\partial}{\partial u} \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} - \frac{\partial}{\partial v} \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \begin{Bmatrix} 22 \\ 1 \end{Bmatrix},$$

$$\frac{D'' - D'^2}{H^2} E = \frac{\partial}{\partial v} \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} - \frac{\partial}{\partial u} \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} + \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}^2,$$

$$\frac{D'' - D'^2}{H^2} G = \frac{\partial}{\partial u} \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} - \frac{\partial}{\partial v} \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}^2,$$

$$\frac{D'' - D'^2}{H^2} F = \frac{\partial}{\partial v} \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} - \frac{\partial}{\partial u} \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} + \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}$$

he expressions for the Christoffel symbols are substituted equations the latter reduce to the single equation

$$\frac{DD'' - D'^2}{H^2} = \frac{1}{2H} \left\{ \frac{\partial}{\partial u} \left[\frac{F}{EH} \frac{\partial E}{\partial v} - \frac{1}{H} \frac{\partial G}{\partial u} \right] + \frac{\partial}{\partial v} \left[\frac{2}{H} \frac{\partial F}{\partial u} - \frac{1}{H} \frac{\partial E}{\partial v} - \frac{F}{EH} \frac{\partial E}{\partial u} \right] \right\}$$

uation was discovered by Gauss, and is called the *Gauss* of condition upon the fundamental functions. The left-member of the equation is the expression for the total curvature of the surface. Hence we have the celebrated theorem of

expression for the total curvature of a surface is a function of the fundamental coefficients of the first order and of their derivatives of the first and second orders

the expressions for c_1 and c_2 are calculated, we find that two of equations (10) are

$$\frac{\partial D}{\partial v} - \frac{\partial D'}{\partial u} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} D + \left(\begin{Bmatrix} 11 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \right) D' + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} D'' = 0,$$

$$\frac{\partial D''}{\partial u} - \frac{\partial D'}{\partial v} + \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} D + \left(\begin{Bmatrix} 22 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \right) D' - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} D'' = 0$$

These are the *Codazzi equations*, so called because they are equivalent to the equations found by Codazzi*, however, it should be mentioned that Mainardi was brought to similar results somewhat earlier† It is sometimes convenient to have these equations written in the form

$$(13') \quad \begin{cases} \frac{\partial}{\partial v} \frac{D}{H} - \frac{\partial}{\partial u} \frac{D'}{H} + \left\{ \begin{smallmatrix} 22 \\ 2 \end{smallmatrix} \right\} \frac{D}{H} - 2 \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\} \frac{D'}{H} + \left\{ \begin{smallmatrix} 11 \\ 2 \end{smallmatrix} \right\} \frac{D''}{H} = 0, \\ \frac{\partial}{\partial u} \frac{D''}{H} - \frac{\partial}{\partial v} \frac{D'}{H} + \left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\} \frac{D}{H} - 2 \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\} \frac{D'}{H} + \left\{ \begin{smallmatrix} 11 \\ 1 \end{smallmatrix} \right\} \frac{D''}{H} = 0, \end{cases}$$

which reduce readily to (13) by means of (3)

With the aid of equations (7) we find that the conditions of integrability of equations (8) and similar ones in Y and Z reduce to (13)

From the preceding theorem and the definition of applicable surfaces (§ 43) follows the theorem

Two applicable surfaces have the same total curvature at corresponding points

As a consequence we have

Every surface applicable to a plane is the tangent surface of a twisted curve

For, when a surface is applicable to a plane its linear element is reducible to $ds^2 = du^2 + dv^2$, and consequently its total curvature is zero at every point by (12) From (IV, 73) it follows that

$$K^2 = \left[\frac{\partial(Y, Z)}{\partial(u, v)} \right]^2 + \left[\frac{\partial(Z, X)}{\partial(u, v)} \right]^2 + \left[\frac{\partial(X, Y)}{\partial(u, v)} \right]^2 = 0$$

Hence X, Y, Z are functions of a single parameter, and therefore the surface is the tangent surface of a twisted curve (cf § 27)

Incidentally we have proved the theorem

When K is zero at all points of a surface the latter is developable, and conversely

* Sulle coordinate curvilinee d una superficie e dello spazio, *Annali*, Ser 3, Vol II (1868) p 269

† *Giornale dell Istituto Lombardo*, Vol IX, p 395

undamental theorem When the lines of curvature are
ric, the Gauss and Codazzi equations (12), (13) reduce to

$$\begin{cases} \frac{DD''}{\sqrt{EG}} + \frac{\partial}{\partial u} \left(\frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \right) = 0, \\ \frac{\partial}{\partial v} \left(\frac{D}{\sqrt{E}} \right) - \frac{D''}{G} \frac{\partial \sqrt{E}}{\partial v} = 0, \\ \frac{\partial}{\partial u} \left(\frac{D''}{\sqrt{G}} \right) - \frac{D}{E} \frac{\partial \sqrt{G}}{\partial u} = 0 \end{cases}$$

ection-cosines of the tangents to the parametric curves,
it and $u = \text{const}$, have the respective values

$$\begin{cases} X_1, & Y_1, & Z_1 = \frac{1}{\sqrt{E}} \frac{\partial x}{\partial u}, & \frac{1}{\sqrt{E}} \frac{\partial y}{\partial u}, & \frac{1}{\sqrt{E}} \frac{\partial z}{\partial u}, \\ X_2, & Y_2, & Z_2 = \frac{1}{\sqrt{G}} \frac{\partial x}{\partial v}, & \frac{1}{\sqrt{G}} \frac{\partial y}{\partial v}, & \frac{1}{\sqrt{G}} \frac{\partial z}{\partial v} \end{cases}$$

ns of equations (7) and (8) we find

$$\begin{aligned} \frac{\partial X_1}{\partial u} &= -\frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} X_2 + \frac{D}{\sqrt{E}} X, & \frac{\partial X_1}{\partial v} &= \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} X_2, \\ \frac{\partial X_2}{\partial u} &= \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} X_1, & \frac{\partial X_2}{\partial v} &= -\frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} X_1 + \frac{D''}{\sqrt{G}} X, \\ \frac{\partial X}{\partial u} &= -\frac{D}{\sqrt{E}} X_1, & \frac{\partial X}{\partial v} &= -\frac{D''}{\sqrt{G}} X_2, \end{aligned}$$

ilar equations obtained by replacing X_1, X_2, X by Y_1, Y_2, Y
vely, and by Z_1, Z_2, Z From (15) we have

$$\begin{cases} x = \int \sqrt{E} X_1 du + \sqrt{G} X_2 dv, \\ y = \int \sqrt{E} Y_1 du + \sqrt{G} Y_2 dv, \\ z = \int \sqrt{E} Z_1 du + \sqrt{G} Z_2 dv \end{cases}$$

roceed to the proof of the converse theorem

*four functions, E, G, D, D'' , satisfying equations (14), there
surface for which $E, 0, G, D, 0, D''$ are the fundamental
es of the first and second order respectively*

In the first place we remark that all the conditions of integrability of the equations (16) are satisfied in consequence of (14). Hence these equations admit sets of particular solutions whose values for the initial values of u and v are arbitrary. From the form of equations (16) it follows (cf § 13) that, if two such sets of particular solutions be denoted by X_1, X_2, X and Y_1, Y_2, Y , then

$$(18) \quad \begin{cases} X_1^2 + X_2^2 + X^2 = \text{const}, \\ Y_1^2 + Y_2^2 + Y^2 = \text{const}, \\ X_1 Y_1 + X_2 Y_2 + XY = \text{const} \end{cases}$$

From the theory of differential equations we know that there exist three particular sets of solutions $X_1, X_2, X, Y_1, Y_2, Y, Z_1, Z_2, Z$, which for the initial values of u and v have the values 1, 0, 0, 0, 1, 0, 0, 0, 1. In this case equations (18) become

$$(19) \quad \begin{cases} X_1^2 + X_2^2 + X^2 = 1, \\ Y_1^2 + Y_2^2 + Y^2 = 1, \\ X_1 Y_1 + X_2 Y_2 + XY = 0, \end{cases}$$

which are true for all values of u and v . In like manner we have

$$(19') \quad \begin{cases} Z_1^2 + Z_2^2 + Z^2 = 1, \\ X_1 Z_1 + X_2 Z_2 + \lambda Z = 0, \\ Y_1 Z_1 + Y_2 Z_2 + YZ = 0 \end{cases}$$

From (16) it follows that the expressions in the right-hand members of (17) are exact differentials, and that the surface defined by these equations has, for its linear element and its second quadratic form, the expressions

$$(20) \quad E du^2 + G dv^2, \quad D du^2 + D'' dv^2$$

respectively

Suppose, now, that we had a second system of three sets of solutions of equations (16) satisfying the conditions (19), (19'). By a motion in space we could make these λ 's, ν 's, and ν 's equal to the corresponding ones of the first system for the initial values of u and v . But then, because of the relations similar to (18), they would be equal for all values of u and v , as shown in § 13. Hence, to within a motion in space, a surface is determined by two quadratic forms (20). As in § 13, it can be shown that the solution of equations (16) reduces to the integration of an equation of Riccati

* we shall find that the direction-cosines of any two per-
lar lines in the tangent plane to a surface, and of the normal
urface, satisfy a system of equations similar in form to (16)
r, these equations possess the property that sets of solu-
nsfy the conditions (18) when the parametric lines are any
r Hence the choice of lines of curvature as parametric
mplies the preceding equations, but the result is a general
nsequently we have the following fundamental theorem

the coefficients of two quadratic forms,

$$du^2 + 2F dudv + G dv^2, \quad D du^2 + 2D' dudv + D'' dv^2,$$

*he equations of Gauss and Codazzi, there exists a surface,
to within its position in space, for which these forms are
ely the first and second fundamental quadratic forms, and
mination of the surface requires the integration of a Riccati
and quadratures*

III, 3), (5) and (6), it follows that if E, F, G, D, D', D'' are the funda-
ntions for a surface of coordinates (x, y, z) , the surface symmetric with
the origin, that is, the surface with the coordinates $(-x, -y, -z)$, has
mental functions $E, F, G, -D, -D', -D''$. Moreover, in consequence
ve theorem, two surfaces whose fundamental quantities bear such a rela-
e moved in space so that they will be symmetric with respect to a point
ces of this kind will be treated as the same surface

EXAMPLES

en the lines of curvature of a surface form an isothermal system, the
said to be *isothermic*. Show that surfaces of revolution are isothermic
w that the hyperbolic paraboloid

$$x = \frac{a}{2}(u + v), \quad y = \frac{b}{2}(u - v), \quad z = \frac{uv}{2}$$

nic

en a surface is isothermic, and the linear element, expressed in terms of
s referring to the lines of curvature, is $ds^2 = \lambda^2(du^2 + dv^2)$, the equations
1 and Gauss are reducible to

$$\frac{\partial}{\partial u} \log \lambda = \frac{\rho_1}{\rho_2} \frac{1}{\rho_1 - \rho_2} \frac{\partial \rho_2}{\partial u}, \quad \frac{\partial}{\partial v} \log \lambda = \frac{\rho_2}{\rho_1} \frac{1}{\rho_2 - \rho_1} \frac{\partial \rho_1}{\partial v},$$

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \log \lambda + \frac{\lambda^2}{\rho_1 \rho_2} = 0$$

d the form of equations (11), (13) when the surface is defined in terms of
ordinates (cf § 39)

† Consult also Scheffers, Vol II pp 310 *et seq* Bianchi Vol I, pp 122-124

5 Show that K is equal to zero for the tangent surface of a twisted curve, taking the linear element of the latter in the form (105), § 20

6 Show that the total curvature of the surface of revolution of the tractrix about its axis is negative and constant

7 Establish the following formulas, in which the differential parameters are formed with respect to the form $Edu^2 + 2Fdu dv + Gdv^2$

$$\Delta_1 X = \frac{X_1^2}{\rho_1^2} + \frac{X_2^2}{\rho_2^2}, \quad \Delta_1(Y, Z) = \frac{Y_1 Z_1}{\rho_1^2} + \frac{Y_2 Z_2}{\rho_2^2},$$

$$\Theta(y, z) = X, \quad \Theta(Y, Z) = \frac{X}{\rho_1 \rho_2},$$

where the quantities have the same significance as in § 65

8 Deduce the identity $\Delta_2 X = \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) X,$

and show therefrom that the curves in which a minimal surface is cut by a family of parallel planes and the orthogonal trajectories of these curves form an isothermal system

66 Fundamental equations in another form We have seen in § 61 that if X, Y, Z denote the direction-cosines of the normal to a surface, the direction-cosines of the normal to the spherical representation of the surface are $\epsilon X, \epsilon Y, \epsilon Z$, where ϵ is ± 1 according as the curvature of the surface is positive or negative. If, then, the second fundamental quantities for the sphere be denoted by $\mathcal{D}, \mathcal{D}', \mathcal{D}''$, we have

$$(21) \quad \mathcal{D} = -\epsilon \mathcal{E}, \quad \mathcal{D}' = -\epsilon \mathcal{F}, \quad \mathcal{D}'' = -\epsilon \mathcal{G},$$

so that for the sphere equations (7) become

$$(22) \quad \begin{cases} \frac{\partial^2 X}{\partial u^2} = \left\{ \begin{smallmatrix} 11 \\ 1 \end{smallmatrix} \right\}' \frac{\partial X}{\partial u} + \left\{ \begin{smallmatrix} 11 \\ 2 \end{smallmatrix} \right\}' \frac{\partial X}{\partial v} - \mathcal{E}X, \\ \frac{\partial^2 X}{\partial u \partial v} = \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \frac{\partial X}{\partial u} + \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \frac{\partial X}{\partial v} - \mathcal{F}X, \\ \frac{\partial^2 X}{\partial v^2} = \left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\}' \frac{\partial X}{\partial u} + \left\{ \begin{smallmatrix} 22 \\ 2 \end{smallmatrix} \right\}' \frac{\partial X}{\partial v} - \mathcal{G}X, \end{cases}$$

where the Christoffel symbols $\left\{ \begin{smallmatrix} rs \\ t \end{smallmatrix} \right\}'$ are formed with respect to the linear element of the spherical representation, namely

$$(23) \quad d\sigma^2 = \mathcal{E} du^2 + 2\mathcal{F} du dv + \mathcal{G} dv^2$$

The conditions of integrability of equations (22) are reducible by means of the latter to

$$A_1 \frac{\partial X}{\partial u} + B_1 \frac{\partial X}{\partial v} = 0, \quad A_2 \frac{\partial X}{\partial u} + B_2 \frac{\partial X}{\partial v} = 0,$$

A_1, A_2, B_1, B_2 are the functions obtained from the quantities b_1, b_2 respectively of § 64 by replacing $\frac{DD''-D'^2}{H^2}, E, F, G$ by $\mathcal{E}, \mathcal{F}, \mathcal{G}$ respectively. Since the above equations must be satisfied by Y and Z , the quantities A_1, A_2, B_1, B_2 must be zero. This is the single equation of condition

$$\frac{1}{H} \left[\frac{\partial}{\partial u} \left(\frac{\mathcal{F}}{H} \frac{\partial \mathcal{E}}{\partial v} - \frac{1}{H} \frac{\partial \mathcal{F}}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{2}{H} \frac{\partial \mathcal{F}}{\partial u} - \frac{1}{H} \frac{\partial \mathcal{E}}{\partial v} - \frac{\mathcal{F}}{H} \frac{\partial \mathcal{E}}{\partial u} \right) \right] = 1$$

ver, the Codazzi equations (13') become, in consequence),

$$\frac{\partial}{\partial v} \left(\frac{\mathcal{E}}{H} \right) - \frac{\partial}{\partial u} \left(\frac{\mathcal{F}}{H} \right) + \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}' \frac{\mathcal{E}}{H} - 2 \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \frac{\mathcal{F}}{H} + \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' \frac{\mathcal{G}}{H} = 0,$$

$$\frac{\partial}{\partial u} \left(\frac{\mathcal{G}}{H} \right) - \frac{\partial}{\partial v} \left(\frac{\mathcal{F}}{H} \right) + \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' \frac{\mathcal{E}}{H} - 2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \frac{\mathcal{F}}{H} + \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}' \frac{\mathcal{G}}{H} = 0,$$

vanish identically

equations (IV, 18) be solved for $\frac{\partial x}{\partial u}$ and $\frac{\partial x}{\partial v}$, we get

$$\begin{cases} \frac{\partial x}{\partial u} = \frac{\mathcal{F}D' - \mathcal{G}D}{H^2} \frac{\partial X}{\partial u} + \frac{\mathcal{F}D - \mathcal{E}D'}{H^2} \frac{\partial X}{\partial v}, \\ \frac{\partial x}{\partial v} = \frac{\mathcal{F}D'' - \mathcal{G}D'}{H^2} \frac{\partial X}{\partial u} + \frac{\mathcal{F}D' - \mathcal{E}D''}{H^2} \frac{\partial X}{\partial v} \end{cases}$$

ans of equations (22) the condition of integrability of these ons, namely

$$\frac{\partial}{\partial v} \left(\frac{\partial x}{\partial u} \right) = \frac{\partial}{\partial u} \left(\frac{\partial x}{\partial v} \right)$$

milar conditions in y and z , reduce to

$$\begin{aligned} \frac{\partial D}{\partial v} - \frac{\partial D'}{\partial u} - \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' D + \left[\left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}' - \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \right] D' + \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' D'' &= 0, \\ \frac{\partial D''}{\partial u} - \frac{\partial D'}{\partial v} + \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' D + \left[\left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}' - \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \right] D' - \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' D'' &= 0 \end{aligned}$$

nce two quadratic forms

$$\mathcal{E} du^2 + 2 \mathcal{F} du dv + \mathcal{G} dv^2, \quad D du^2 + 2 D' du dv + D'' dv^2,$$

, coefficients satisfy the conditions (24), (27), may be taken
linear element of the spherical representation of a surface
s the second quadratic form of the latter. When X, Y, Z are

known, the cartesian coordinates of the surface can be found by quadratures (26), however, the determination of the former requires the solution of a Riccati equation

If the equations

$$D = - \sum \frac{\partial x}{\partial u} \frac{\partial X}{\partial u}, \quad D' = - \sum \frac{\partial x}{\partial u} \frac{\partial X}{\partial v} = - \sum \frac{\partial x}{\partial v} \frac{\partial X}{\partial u}, \quad D'' = - \sum \frac{\partial x}{\partial v} \frac{\partial X}{\partial v},$$

be differentiated with respect to u and v , the resulting equations may be reduced by means of (7) and (22) to the form *

$$28) \quad \left\{ \begin{array}{l} \frac{\partial D}{\partial u} = \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\} D + \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} D' + \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}' D + \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' D', \\ \frac{\partial D}{\partial v} = \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} D + \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} D' + \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' D + \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' D', \\ \frac{\partial D'}{\partial u} = \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\} D' + \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} D'' + \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' D + \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' D', \\ \frac{\partial D'}{\partial v} = \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} D' + \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} D'' + \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' D + \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}' D', \\ \frac{\partial D'}{\partial u} = \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} D + \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} D' + \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}' D' + \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' D', \\ \frac{\partial D'}{\partial v} = \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} D + \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} D' + \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' D' + \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' D', \\ \frac{\partial D'}{\partial u} = \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} D' + \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} D'' + \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' D + \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' D'', \\ \frac{\partial D''}{\partial v} = \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} D' + \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} D'' + \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' D' + \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}' D'' \end{array} \right.$$

67 Tangential coordinates Mean evolute A surface may be looked upon not only as the locus of a point whose position depends upon two parameters, but also as the envelope of its tangent planes. This family of planes depends upon one or two parameters according as the surface is developable or not. We considered the former case in § 27, and now take up the latter.

If W denotes the algebraic distance from the origin to the tangent plane to a surface S at the point $M(x, y, z)$, then

$$(29) \quad W = x\lambda + y\lambda' + z\lambda''$$

If this equation is differentiated with respect to u and v , the resulting equations are reducible, in consequence of (IV, 3), to

$$(30) \quad \frac{\partial W}{\partial u} = \sum x \frac{\partial \lambda}{\partial u}, \quad \frac{\partial W}{\partial v} = \sum x \frac{\partial \lambda}{\partial v}$$

ee equations (29), (30) are linear in x, y, z , and in consequence of (IV, 79, 80) their determinant is equal to $\epsilon/\mathcal{H}$. Hence

$$x = \frac{\epsilon}{\mathcal{H}} \begin{vmatrix} W & Y & Z \\ \frac{\partial W}{\partial u} & \frac{\partial Y}{\partial u} & \frac{\partial Z}{\partial u} \\ \frac{\partial W}{\partial v} & \frac{\partial Y}{\partial v} & \frac{\partial Z}{\partial v} \end{vmatrix},$$

ilar expressions for y and z . From (IV, 11) we deduce the

$$\begin{cases} Y \frac{\partial Z}{\partial u} - Z \frac{\partial Y}{\partial u} = \frac{\epsilon}{\mathcal{H}} \left(-\mathcal{S} \frac{\partial X}{\partial u} + \mathcal{E} \frac{\partial X}{\partial v} \right), \\ Y \frac{\partial Z}{\partial v} - Z \frac{\partial Y}{\partial v} = \frac{\epsilon}{\mathcal{H}} \left(-\mathcal{S} \frac{\partial X}{\partial v} + \mathcal{S} \frac{\partial X}{\partial u} \right) \end{cases}$$

ns of these equations the above expression for x is

$$\mathcal{H}X + \frac{1}{\mathcal{H}^2} \left[\mathcal{S} \frac{\partial W}{\partial u} \frac{\partial X}{\partial u} - \mathcal{S} \left(\frac{\partial W}{\partial u} \frac{\partial X}{\partial v} + \frac{\partial W}{\partial v} \frac{\partial X}{\partial u} \right) + \mathcal{E} \frac{\partial W}{\partial v} \frac{\partial X}{\partial v} \right]$$

we have

$$= WX + \Delta'_1(W, X), \quad y = WY + \Delta'_1(W, Y), \quad z = WZ + \Delta'_1(W, Z),$$

ential parameters being formed with respect to (23)

ersely, if we have four functions X, Y, Z, W of u and v , at the first three satisfy the identity

$$X^2 + Y^2 + Z^2 = 1,$$

is (32) define the surface for which X, Y, Z are the direction- of the tangent plane, and W is the distance of the latter from in. For, from (33), we have

$$\sum X \frac{\partial X}{\partial u} = 0, \quad \sum X \frac{\partial X}{\partial v} = 0,$$

quence of which and formulas (22) we find from (32) that

$$\sum X \frac{\partial x}{\partial u} = 0, \quad \sum X \frac{\partial x}{\partial v} = 0$$

er, equation (29) also follows from (32). Hence a surface letely defined by the functions X, Y, Z, W , which are he *tangential coordinates* of the surface *

When equations (30) are differentiated, we obtain

$$\begin{aligned}\frac{\partial^2 W}{\partial u^2} &= -D + \sum x \frac{\partial^2 X}{\partial u^2}, & \frac{\partial^2 W}{\partial u \partial v} &= -D' + \sum x \frac{\partial^2 X}{\partial u \partial v}, \\ \frac{\partial^2 W}{\partial v^2} &= -D'' + \sum x \frac{\partial^2 X}{\partial v^2}\end{aligned}$$

By means of (22), (29), and (30) these equations are reducible to

$$(34) \quad \begin{cases} D = -\left[\frac{\partial^2 W}{\partial u^2} - \left\{ \begin{smallmatrix} 11 \\ 1 \end{smallmatrix} \right\}' \frac{\partial W}{\partial u} - \left\{ \begin{smallmatrix} 11 \\ 2 \end{smallmatrix} \right\}' \frac{\partial W}{\partial v} + \mathcal{E} W \right], \\ D' = -\left[\frac{\partial^2 W}{\partial u \partial v} - \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \frac{\partial W}{\partial u} - \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \frac{\partial W}{\partial v} + \mathcal{F} W \right], \\ D'' = -\left[\frac{\partial^2 W}{\partial v^2} - \left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\}' \frac{\partial W}{\partial u} - \left\{ \begin{smallmatrix} 22 \\ 2 \end{smallmatrix} \right\}' \frac{\partial W}{\partial v} + \mathcal{G} W \right] \end{cases}$$

When these expressions for D , D' , D'' are substituted in the expression (IV, 77) for $\rho_1 + \rho_2$, the latter becomes

$$\begin{aligned}\rho_1 + \rho_2 &= -\frac{1}{K^2} \left[\mathcal{E} \frac{\partial^2 W}{\partial v^2} - 2 \mathcal{F} \frac{\partial^2 W}{\partial u \partial v} + \mathcal{G} \frac{\partial^2 W}{\partial u^2} \right] - 2W \\ &\quad + \frac{1}{K^2} \left[\left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\}' \mathcal{E} - 2 \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \mathcal{F} + \left\{ \begin{smallmatrix} 11 \\ 1 \end{smallmatrix} \right\}' \mathcal{G} \right] \frac{\partial W}{\partial u} \\ &\quad + \frac{1}{K^2} \left[\left\{ \begin{smallmatrix} 22 \\ 2 \end{smallmatrix} \right\}' \mathcal{E} - 2 \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \mathcal{F} + \left\{ \begin{smallmatrix} 11 \\ 2 \end{smallmatrix} \right\}' \mathcal{G} \right] \frac{\partial W}{\partial v}\end{aligned}$$

By means of (25) this equation can be written in the form

$$(35) \quad \rho_1 + \rho_2 = -(\Delta'_2 W + 2W),$$

where the differential parameter is formed with respect to the linear element (23) of the sphere

Moreover, if $\Delta'_{22}\theta$ denotes the following expression,

$$(36) \quad \Delta'_{22}\theta = \frac{1}{\mathcal{E}\mathcal{G} - \mathcal{F}^2} \left[\left(\frac{\partial^2 \theta}{\partial u^2} - \left\{ \begin{smallmatrix} 11 \\ 1 \end{smallmatrix} \right\}' \frac{\partial \theta}{\partial u} - \left\{ \begin{smallmatrix} 11 \\ 2 \end{smallmatrix} \right\}' \frac{\partial \theta}{\partial v} \right) \right. \\ \left. \left(\frac{\partial^2 \theta}{\partial v^2} - \left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\}' \frac{\partial \theta}{\partial u} - \left\{ \begin{smallmatrix} 22 \\ 2 \end{smallmatrix} \right\}' \frac{\partial \theta}{\partial v} \right) \right. \\ \left. - \left(\frac{\partial^2 \theta}{\partial u \partial v} - \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \frac{\partial \theta}{\partial u} - \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \frac{\partial \theta}{\partial v} \right)^2 \right],$$

it follows from (34) that

$$(37) \quad \rho_1 \rho_2 = \frac{DD'' - D'^2}{\mathcal{E}\mathcal{G} - \mathcal{F}^2} = \Delta'_{22} W + W \Delta'_2 W + W^2$$

using we shall prove that $\Delta_{22}\theta$ is a differential parameter and that it is expressible in the form

$$\Delta_{22}\theta = \frac{2\Delta_2\theta\Delta_1(\theta, \Delta_1\theta) - \Delta_1\Delta_1\theta}{4\Delta_1\theta}$$

at loss of generality we take

$$Edu^2 + Gdv^2$$

quadratic form, with respect to which these differential operators are formed. Then

$$\begin{aligned} \frac{1}{E}, \quad \Delta_2 u &= \frac{1}{2E} \left(\frac{1}{G} \frac{\partial G}{\partial u} - \frac{1}{E} \frac{\partial E}{\partial u} \right), \quad \Delta_1(u, \Delta_1 u) = -\frac{1}{E^2} \frac{\partial E}{\partial u} \\ &= \frac{E \left(\frac{\partial E}{\partial v} \right)^2 + G \left(\frac{\partial E}{\partial u} \right)^2}{E^2 G}, \quad \Delta_{22} u = -\frac{1}{4E^2 G} \left[\frac{\partial E}{\partial u} \frac{\partial G}{\partial u} + \left(\frac{\partial E}{\partial v} \right)^2 \right] \end{aligned}$$

substitution we find

$$\Delta_{22} u = \frac{2\Delta_2 u \Delta_1(u, \Delta_1 u) - \Delta_1 \Delta_1 u}{4\Delta_1 u}$$

the terms in the right-hand member are differential parameter values which are independent of the choice of parameters, in terms of which (39) is expressed. Hence equation (38) is an identity.

The coordinates x_0, y_0, z_0 of the point on the normal to a surface midway between the centers of principal curvature have the following expressions:

$$x_0 = x + \frac{1}{2}(\rho_1 + \rho_2)X, \quad y_0 = y + \frac{1}{2}(\rho_1 + \rho_2)Y, \quad z_0 = z + \frac{1}{2}(\rho_1 + \rho_2)Z$$

the surface S_0 enveloped by the plane through this point, which is parallel to the tangent plane to the given surface, is called the *evolute* of the latter.

If W denotes the distance from the origin to this plane, we have

$$W_0 = \sum \lambda x_0 = W + \frac{1}{2}(\rho_1 + \rho_2)$$

Equation (35) thus may be written

$$W_0 = -\frac{1}{2} \Delta_2' W$$

EXAMPLES

1 Derive the equations of the lines of curvature and the expressions for the principal radii in terms of W , when the parametric lines on the sphere are

- (i) meridians and parallels,
(ii) the imaginary generators

Show that in the latter case the curves corresponding to the generators lie symmetrically with respect to the lines of curvature

2 Let W_1 and W_2 denote the distances from the origin to the planes through the normal to a surface and the tangents to the lines of curvature $v = \text{const}$, $u = \text{const}$ respectively, so that we have

$$W_1 = xX_1 + yY_1 + zZ_1, \quad W_2 = xX_2 + yY_2 + zZ_2$$

Show that

$$\Delta_1 W = \frac{W_1^2}{\rho_1^2} + \frac{W_2^2}{\rho_2^2}, \quad \Delta_1(W, x) = -\frac{X_1 W_1}{\rho_1} - \frac{X_2 W_2}{\rho_2},$$

$$\Delta_1(W, X) = \frac{X_1 W_1}{\rho_1^2} + \frac{X_2 W_2}{\rho_2^2}, \quad \Theta(W, x) = -\frac{X_2 W_1}{\rho_1} + \frac{X_1 W_2}{\rho_2},$$

the differential parameters being formed with respect to $E du^2 + 2 F du dv + G dv^2$

3 If $2q = x^2 + y^2 + z^2$, then we have

$$\Delta_1 q = 2q - W^2, \quad \Delta_2 q = 2 + W \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right), \quad \Delta_1(q, W) = - \left(\frac{W_1^2}{\rho_1} + \frac{W_2^2}{\rho_2} \right)$$

4 Show that when the lines of curvature are parametric

$$-\frac{1}{\rho_1} \frac{\partial q}{\partial u} = \frac{\partial W}{\partial u}, \quad -\frac{1}{\rho_2} \frac{\partial q}{\partial v} = \frac{\partial W}{\partial v}$$

5 The determination of surfaces whose mean evolute is a point is the same problem as finding isothermal systems of lines on the sphere

68 The moving trihedral The fundamental equations of condition may be given another form, in which they are frequently used by French writers. In deriving them we refer the surface to a moving set of rectangular axes called the trihedral T . Its vertex M is a point of the surface, the xy -plane is tangent to the surface at M , and the positive z axis coincides with the positive direction of the normal to the surface at M . The position of the x - and y -axes is determined by the angle U which the tangent to the curve $v = \text{const}$ through M makes with the x -axis, U being a given function of u and v .

In Chapter I we considered another moving trihedral, consisting of the tangent, principal normal, and binormal of a twisted curve. Let us associate such a trihedral with the curve $v = \text{const}$ through

all it the trihedral t_u . We have found (§ 16) that the variation of the direction-cosines a' , b' , c' of a line L , fixed in space, with reference to t_u , as its vertex moves along the curve which C_u , are given by

$$\frac{da'}{ds_u} = \frac{b'}{\rho_u}, \quad \frac{db'}{ds_u} = -\left(\frac{a'}{\rho_u} + \frac{c'}{\tau_u}\right), \quad \frac{dc'}{ds_u} = \frac{b'}{\tau_u},$$

where ρ_u , τ_u denote the radii of first and second curvature of C_u , ds_u its linear element, evidently the latter may be replaced by du

the direction-cosines of L with respect to the trihedral T have

$$\begin{cases} a = a' \cos U - (b' \sin \bar{\omega}_u - c' \cos \bar{\omega}_u) \sin U, \\ b = a' \sin U + (b' \sin \bar{\omega}_u - c' \cos \bar{\omega}_u) \cos U, \\ c = b' \cos \bar{\omega}_u + c' \sin \bar{\omega}_u, \end{cases}$$

$\bar{\omega}_u$ is the angle which the positive direction of the z -axis makes with the positive direction of the principal normal to C_u at M , the angle being measured toward the positive direction of the binormal τ_u . From equations (42) and (43) we obtain the following

$$\frac{\partial a}{\partial u} = br - cq, \quad \frac{\partial b}{\partial u} = cp - ar, \quad \frac{\partial c}{\partial u} = aq - bp,$$

where p , q , r have the following significance

$$\begin{cases} p = \sqrt{E} \left[\cos U \left(\frac{d\bar{\omega}_u}{ds_u} - \frac{1}{\tau_u} \right) + \sin U \frac{\cos \bar{\omega}_u}{\rho_u} \right], \\ q = \sqrt{E} \left[\sin U \left(\frac{d\bar{\omega}_u}{ds_u} - \frac{1}{\tau_u} \right) - \cos U \frac{\cos \bar{\omega}_u}{\rho_u} \right], \\ r = \sqrt{E} \left(\frac{\sin \bar{\omega}_u}{\rho_u} - \frac{dU}{ds_u} \right) \end{cases}$$

In like manner, we consider the trihedral t_v of the curve C_v passing through M , denoted by C_v , we obtain the equations

$$\frac{\partial a}{\partial v} = br_1 - cq_1, \quad \frac{\partial b}{\partial v} = cp_1 - ar_1, \quad \frac{\partial c}{\partial v} = aq_1 - bp_1,$$

where p_1 , q_1 , r_1 can be obtained from (45) by replacing $\sqrt{E}$, U , s_u , $\bar{\omega}_u$ by $\sqrt{G}$, V , s_v , $\bar{\omega}_v$, ρ_v , τ_v . As V denotes the angle which the z -axis to the curve C_v at M makes with the x -axis, we have

$$V - U = \omega$$

oves along a curve other than a parametric curve determined by a value of dv/du , the re evidently given by

$$, \quad \frac{\partial b}{\partial u} \frac{du}{ds} + \frac{\partial b}{\partial v} \frac{dv}{ds}, \quad \frac{\partial c}{\partial u} \frac{du}{ds} + \frac{\partial c}{\partial v} \frac{dv}{ds},$$

fferential quotients have the above values

iental equations of condition Suppose that we asso-
trihedral T a second trihedral T_0 whose vertex O is
, about which it revolves in such a manner that its
ays parallel to the corresponding edges of T , as the
latter moves over the surface in a given manner
f T_0 is completely determined by the nine direction-
edges with three mutually perpendicular lines $L_1, L_2,$
 L_3 through O Call these direction-cosines $a_1, b_1, c_1, a_2, b_2, c_2, a_3,$
 b_3, c_3 These functions must satisfy the equations

$$(47) \quad \begin{cases} \frac{\partial a}{\partial u} = br - cq, & \frac{\partial b}{\partial u} = cp - ar, & \frac{\partial c}{\partial u} = aq - bp, \\ \frac{\partial a}{\partial v} = br_1 - cq_1, & \frac{\partial b}{\partial v} = cp_1 - ar_1, & \frac{\partial c}{\partial v} = aq_1 - bp_1 \end{cases}$$

If we equate the two values of $\frac{\partial^2 a}{\partial u \partial v}$, obtained from the first two of these equations, and in the reduction of the resulting equation make use of (47), we find

$$b \left(\frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u} - pq_1 + qp_1 \right) = c \left(\frac{\partial q}{\partial v} - \frac{\partial q_1}{\partial u} - rp_1 + pr_1 \right)$$

Since this equation must be true when b and c have the values $b_1, c_1, b_2, c_2, b_3, c_3$, the expressions in parenthesis must be equal to zero Proceeding in the same manner with $\frac{\partial^2 b}{\partial u \partial v}$ and $\frac{\partial^2 c}{\partial u \partial v}$, we obtain the following fundamental equations *

$$(48) \quad \begin{cases} \frac{\partial p}{\partial v} - \frac{\partial p_1}{\partial u} = qr_1 - rq_1, \\ \frac{\partial q}{\partial v} - \frac{\partial q_1}{\partial u} = rp_1 - pr_1, \\ \frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u} = pq_1 - qp_1 \end{cases}$$

* These equations were first obtained by Combescore *Annales de l'École Normale*, Ser 1, Vol IV (1867), p 108, cf also Darboux, Vol I, p 48

necessary conditions upon the six functions p , q , r , in at the nine functions a_1 , a_2 , c_3 may determine the position of the trihedral T_0 , are also sufficient conditions. The proof of this is due to that given in § 65.

Formulas (47) have been obtained by Darboux * from a study of the notion of the trihedral T_0 . He has called p , q , r , r_1 the

return to the consideration of the moving trihedrals T and (x, y, z) and (x', y', z') denote the coordinates of a point P with respect to T and t_u respectively. Between these coordinates the following relations hold:

$$\begin{cases} x = x' \cos U - (y' \sin \bar{\omega}_u - z' \cos \bar{\omega}_u) \sin U, \\ y = x' \sin U + (y' \sin \bar{\omega}_u - z' \cos \bar{\omega}_u) \cos U, \\ z = y' \cos \bar{\omega}_u + z' \sin \bar{\omega}_u \end{cases}$$

the absolute increments of P with respect to the trihedral t_u at M be indicated by δ , and increments relative to the moving axes by d , we have, from § 16,

$$\frac{\delta x'}{\delta s_u} = \frac{dx'}{ds_u} - \frac{y'}{\rho_u} + 1, \quad \frac{\delta y'}{\delta s_u} = \frac{dy'}{ds_u} + \frac{x'}{\rho_u} + \frac{z'}{\tau_u}, \quad \frac{\delta z'}{\delta s_u} = \frac{dz'}{ds_u} - \frac{y'}{\tau_u}$$

From (49), (50), and (45) we obtain the following †

$$\frac{\delta x}{\partial u} = \frac{dx}{\partial u} + \sqrt{E} \cos U - ry + qz,$$

$$\frac{\delta y}{\partial u} = \frac{dy}{\partial u} + \sqrt{E} \sin U - pz + rx,$$

$$\frac{\delta z}{\partial u} = \frac{dz}{\partial u} - qx + py$$

Relations similar to these follow also from the consideration of the trihedral t_v . Hence, when the trihedral T moves over the surface with its vertex M describing a curve determined by a function $v = v(u)$, the increments of the coordinates of a point $P(x, y, z)$, in the directions of the axes of the trihedral, in the

* Vol I, chaps I and V.
 † In deriving these equations we have made use of the fact that equations (49) define the determination of coordinates, and consequently hold when the coordinates are replaced by the projections of an absolute displacement of P .

absolute displacement of P , which may also be moving relative to these axes, have the values *

$$(51) \quad \begin{cases} \delta x = dx + \xi du + \xi_1 dv + (q du + q_1 dv)z - (r du + r_1 dv)y, \\ \delta y = dy + \eta du + \eta_1 dv + (r du + r_1 dv)x - (p du + p_1 dv)z, \\ \delta z = dz + (p du + p_1 dv)y - (q du + q_1 dv)x, \end{cases}$$

where we have put

$$(52) \quad \begin{cases} \xi = \sqrt{E} \cos U, & \eta = \sqrt{E} \sin U, \\ \xi_1 = \sqrt{G} \cos V = \sqrt{G} \cos(\omega + U), & \eta_1 = \sqrt{G} \sin V = \sqrt{G} \sin(\omega + U) \end{cases}$$

The coordinates of M are $(0, 0, 0)$, so that the increments of its displacements are

$$(53) \quad \delta x = \xi du + \xi_1 dv, \quad \delta y = \eta du + \eta_1 dv, \quad \delta z = 0$$

If (x_1, y_1, z_1) denote the coordinates of M with respect to the fixed axes formed by the lines L_1, L_2, L_3 previously defined, it follows that

$$(54) \quad \frac{\partial x_1}{\partial u} = \xi a_1 + \eta b_1, \quad \frac{\partial x_1}{\partial v} = \xi_1 a_1 + \eta_1 b_1,$$

and similar expressions for y_1 and z_1 , where $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$ are the direction-cosines of the fixed axes with reference to the moving axes. Since the latter satisfy equations (47), the conditions that the two values of $\frac{\partial^2 x_1}{\partial u \partial v}$ obtained from (54) be equal, and similarly for $\frac{\partial^2 y_1}{\partial u \partial v}$ and $\frac{\partial^2 z_1}{\partial u \partial v}$, are

$$(55) \quad \begin{cases} \frac{\partial \xi}{\partial v} - \frac{\partial \xi_1}{\partial u} = \eta r_1 - \eta_1 r, \\ \frac{\partial \eta}{\partial v} - \frac{\partial \eta_1}{\partial u} = \xi_1 r - \xi r_1, \\ \eta_1 p - \eta p_1 + \xi q_1 - \xi_1 q = 0 \end{cases}$$

When we have ten functions $\xi, \xi_1, \eta, \eta_1, p, p_1, q, q_1, r, r_1$, satisfying these conditions and (48), the functions a_1, b_1, c_1 can be found by the solution of a Riccati equation, and x_1, y_1, z_1 by quadratures. Hence equations (48) and (55) are sufficient as well as necessary, and consequently are equivalent to the Gauss and Codazzi equations

Linear element Lines of curvature From (53) we see that near element of the surface is

$$ds^2 = (\xi du + \xi_1 dv)^2 + (\eta du + \eta_1 dv)^2$$

a necessary and sufficient condition that the parametric be orthogonal is

$$\xi\xi_1 + \eta\eta_1 = 0$$

a sphere of radius c the coordinates of the center are $(0, 0, -c)$, it being understood that the positive normal is directed outwards. As this is a fixed point, it follows from equations (51) that whatever be the value of dv/du we must have

$$\xi du + \xi_1 dv - (q du + q_1 dv)c = 0,$$

hence

$$\eta du + \eta_1 dv + (p du + p_1 dv)c = 0,$$

$$\frac{\xi}{q} = \frac{\eta}{-p} = \frac{\xi_1}{q_1} = \frac{\eta_1}{-p_1} = c$$

hence, when these equations are satisfied, the point $(0, 0, -c)$ is fixed in space, and therefore the surface is a sphere. Moreover, suppose that we have a proportion as (58), where the factor of proportionality is not necessarily constant. We may then call it t . When the values from (58) are substituted in (55) and on is made in accordance with (48) we get

$$\eta \frac{\partial t}{\partial v} - \eta_1 \frac{\partial t}{\partial u} = 0, \quad \xi \frac{\partial t}{\partial v} - \xi_1 \frac{\partial t}{\partial u} = 0$$

t is constant unless $\xi\eta_1 - \xi_1\eta$ is zero, which, from (58) and § 81, is seen to be possible only in case the surface is isotropic developable.

definition (§ 51) a line of curvature is a curve along which normals to the surface form a developable surface. When the surface is displaced along one of these lines, a point $(0, 0, \rho)$ must move in such a way that δx and δy are zero. Hence we must have

$$\xi du + \xi_1 dv + (q du + q_1 dv)\rho = 0,$$

$$\eta du + \eta_1 dv - (p du + p_1 dv)\rho = 0$$

Eliminating ρ and dv/du respectively, we obtain the equation of lines of curvature,

$$\xi du + \xi_1 dv)(p du + p_1 dv) + (\eta du + \eta_1 dv)(q du + q_1 dv) = 0,$$

the equation of the principal radii,

$$\rho^2(pq_1 - qp_1) + \rho(q\eta_1 - q_1\eta + p\xi_1 - p_1\xi) + (\xi\eta_1 - \eta\xi_1) = 0$$

From (59) it follows that a necessary and sufficient condition for the parametric lines to be the lines of curvature is

$$\xi p + \eta q = 0, \quad \xi_1 p_1 + \eta_1 q_1 = 0$$

We may replace these equations by

$$p = \lambda \eta, \quad q = -\lambda \xi, \quad p_1 = \lambda_1 \eta_1, \quad q_1 = -\lambda_1 \xi_1,$$

thus introducing two auxiliary functions λ and λ_1 . When these values are substituted in the third of (55), we have

$$(\lambda - \lambda_1)(\xi \xi_1 + \eta \eta_1) = 0$$

If λ and λ_1 are equal, the above equations are of the form (58), which were seen to be characteristic of the sphere and the isotropic developable. Hence the second factor is zero, so that equations (61) may be replaced by

$$(62) \quad \xi \xi_1 + \eta \eta_1 = 0, \quad p p_1 + q q_1 = 0,$$

or

$$(63) \quad \xi_1 = \eta = 0, \quad p = q_1 = 0$$

From (52) it follows that in the latter case the x - and y -axes are tangent to the curves $v = \text{const}$ and $u = \text{const}$. We shall consider this case later.

From (60) and (52) we find that the expression for the total curvature of the surface is

$$K = \frac{1}{\rho_1 \rho_2} = \frac{p q_1 - p_1 q}{\xi \eta_1 - \xi_1 \eta} = \frac{p q_1 - p_1 q}{\sqrt{E G \sin \omega}},$$

where ω denotes the angle between the parametric curves. Hence the third of equations (48) may be written

$$(64) \quad \frac{\sqrt{E G \sin \omega}}{\rho_1 \rho_2} = \frac{H}{\rho_1 \rho_2} = \frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u}$$

71 Conjugate directions and asymptotic directions **Spherical representation** We have found (§ 54) that the direction in the tangent plane conjugate to a given direction is the characteristic of this plane as it envelopes the surface in the given direction. Hence, from the point of view of the moving trihedral, the direction conjugate to a displacement, determined by a value of dv/du , is the line in the xy -plane which passes through the origin, and which does not experience an absolute displacement in the direction of the z axis. From the third of equations (51) it is seen that the equation of this line is

$$(65) \quad (p du + p_1 dv) y - (q du + q_1 dv) x = 0$$

increments of u and v , corresponding to a displacement in the direction of this line, be indicated by d_1u and d_1v , the quantities x and y are proportional to $(\xi d_1u + \xi_1 d_1v)$ and $(\eta d_1u + \eta_1 d_1v)$

and y in (65) are replaced by these values, the resulting equation may be reduced to

$$p\eta - q\xi) dud_1u + (p\eta_1 - q\xi_1) dud_1v + (p_1\eta - q_1\xi) d_1u dv \\ + (p_1\eta_1 - q_1\xi_1) d_1v dv = 0$$

As a consequence of (55) the coefficients of dud_1v and $d_1u dv$ are equal so that the equation is symmetrical with respect to the order of differentials, thus establishing the fact that the relation between a line and its conjugate is reciprocal

Under the condition that the parametric lines be conjugate, equation (66) is satisfied by $du = 0$ and $d_1v = 0$. Hence we must have

$$p\eta_1 - q\xi_1 = 0, \quad p_1\eta - q_1\xi = 0$$

It should be noticed that equations (61) are a consequence of the conditions (62) and (67). Hence we have the result that the lines of curvature form the only orthogonal conjugate system

From (66) it follows that the asymptotic directions are given by

$$\eta - q\xi) du^2 + (p\eta_1 - q\xi_1 + p_1\eta - q_1\xi) dudv + (p_1\eta_1 - q_1\xi_1) dv^2 = 0$$

The spherical representation of a surface is traced out by the point T , whose coordinates are $(0, 0, 1)$ with respect to the tri- T_0 of fixed vertex. From (51) we find that the projections of the displacement of m , corresponding to a displacement along the line, are

$$X = q du + q_1 dv, \quad \delta Y = -(p du + p_1 dv), \quad \delta Z = 0$$

The linear element of the spherical representation is

$$d\sigma^2 = (q du + q_1 dv)^2 + (p du + p_1 dv)^2$$

The line defined by (65) is evidently perpendicular to the direction of the displacement of m , as given by (69). Hence the tangent to the spherical representation of a curve upon a surface is perpendicular to the direction conjugate to the curve at the corresponding point. Therefore the tangents to a line of curvature and its representation are parallel, whereas an asymptotic direction and its representation are perpendicular (§ 61)



72 Fundamental relations and formulas From equations (53) and (69) we have, for the point M on the surface,

$$(71) \quad \begin{cases} \frac{\delta x}{\delta u} = \xi, & \frac{\delta y}{\delta u} = \eta, & \frac{\delta z}{\delta u} = 0, \\ \frac{\delta x}{\delta v} = \xi_1, & \frac{\delta y}{\delta v} = \eta_1, & \frac{\delta z}{\delta v} = 0, \end{cases}$$

and

$$(72) \quad \begin{cases} \frac{\delta X}{\delta u} = q, & \frac{\delta Y}{\delta u} = -p, & \frac{\delta Z}{\delta u} = 0, \\ \frac{\delta X}{\delta v} = q_1, & \frac{\delta Y}{\delta v} = -p_1, & \frac{\delta Z}{\delta v} = 0 \end{cases}$$

Consequently the following relations hold between the fundamental coefficients, the rotations, and the translations

$$(73) \quad \begin{cases} E = \xi^2 + \eta^2, & F = \xi\xi_1 + \eta\eta_1, & G = \xi_1^2 + \eta_1^2, \\ D = p\eta - q\xi, & D' = p_1\eta - q_1\xi = p\eta_1 - q\xi_1, & D'' = p_1\eta_1 - q_1\xi_1, \\ \mathcal{E} = p^2 + q^2, & \mathcal{F} = pp_1 + qq_1, & \mathcal{G} = p_1^2 + q_1^2 \end{cases}$$

When, in particular, the parametric system on a surface is orthogonal, and the x - and y -axes of the trihedral are tangent to the curves $v = \text{const}$ and $u = \text{const}$ through the vertex, equations (52) are

$$(74) \quad \xi = \sqrt{E}, \quad \eta = \xi_1 = 0, \quad \eta_1 = \sqrt{G},$$

and equations (55) reduce to

$$(75) \quad r = -\frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v}, \quad r_1 = \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u}, \quad p\sqrt{G} + q_1\sqrt{E} = 0$$

Moreover, equations (45) and the similar ones for p , q , r become

$$(76) \quad \begin{cases} p = \sqrt{E} \left(\frac{d\omega_u}{ds} - \frac{1}{\tau_u} \right), & q = -\sqrt{E} \frac{\cos \omega_u}{\rho_u}, & r = \sqrt{E} \frac{\sin \omega_u}{\rho}, \\ p_1 = \sqrt{G} \frac{\cos \omega_v}{\rho_v}, & q_1 = \sqrt{G} \left(\frac{d\omega_v}{ds} - \frac{1}{\tau_v} \right), & r_1 = \sqrt{G} \frac{\sin \omega_v}{\rho_v} \end{cases}$$

The first two of equations (75) lead, by means of (76), to

$$(77) \quad \frac{\sin \omega_u}{\rho_u} = -\frac{1}{\sqrt{E} \tau_u} \frac{\partial \sqrt{E}}{\partial v}, \quad \frac{\sin \omega_v}{\rho_v} = \frac{1}{\sqrt{G} \tau_v} \frac{\partial \sqrt{G}}{\partial u},$$

which follow also from § 58

The third of equations (75) establishes the fact, previously remarked in § 59, that the geodesic torsion in two orthogonal directions differs only in sign

direction-cosines X_1, Y_1, Z_1 of the tangent to the curve are represented by the motion of the point $(1, 0, 0)$ of the trihedral T_0 vertex. From (51) we have

$$\begin{cases} \frac{\delta X_1}{\delta u} = 0, & \frac{\delta Y_1}{\delta u} = r, & \frac{\delta Z_1}{\delta u} = -q, \\ \frac{\delta X_1}{\delta v} = 0, & \frac{\delta Y_1}{\delta v} = r_1, & \frac{\delta Z_1}{\delta v} = -q_1 \end{cases}$$

these equations we see that as a point describes a curve $v = \text{const.}$, namely tangent to this curve undergoes an infinitesimal rotation consisting of two parts, one in amount $r du$ about the normal to the surface and the other, about the line in the tangent plane perpendicular to the tangent to C_u . Namely, by their definition, the geodesic and normal curvature of C_u are $1 - q/\sqrt{E}$ respectively. Moreover, it is seen from (72) that as a point C_u the normal to the surface undergoes a rotation consisting of the component $r du$ about the line in the tangent plane perpendicular to the tangent, and about the tangent. Hence, if C_u were a geodesic, the torsion would be within the sign at least. Thus by geometrical considerations we have the fundamental relations (76)

suppose now that the parametric system is any whatever. By the definition of the differential parameters (§ 37) it follows

$$E = H^2 \Delta_1 v, \quad G = H^2 \Delta_1 u$$

namely if P, Q, R denote functions similar to p, q, r , for a curve

$$\phi(u, v) = \text{const.},$$

which passes through M and whose tangent makes the angle Φ with the moving x -axis, we have, from (45),

$$\begin{cases} P = H_1 \sqrt{\Delta_1 \phi} \left[\cos \Phi \left(\frac{d\bar{\omega}}{ds} - \frac{1}{\tau} \right) + \sin \Phi \frac{\cos \bar{\omega}}{\rho} \right], \\ Q = H_1 \sqrt{\Delta_1 \phi} \left[\sin \Phi \left(\frac{d\bar{\omega}}{ds} - \frac{1}{\tau} \right) - \cos \Phi \frac{\cos \bar{\omega}}{\rho} \right], \\ R = H_1 \sqrt{\Delta_1 \phi} \left[\frac{\sin \bar{\omega}}{\rho} - \frac{d\Phi}{ds} \right], \end{cases}$$

by (III, 51) $H_1^{-2} = \Delta_1 \phi \Delta_1 \psi - \Delta_1^2(\phi, \psi)$ and $\psi = \text{const.}$ defines a family of curves

over, equations analogous to (44) are

$$\frac{da}{ds} = \frac{bR - cQ}{H_1 \sqrt{\Delta_1 \phi}}, \quad \frac{db}{ds} = \frac{cP - aR}{H_1 \sqrt{\Delta_1 \phi}}, \quad \frac{dc}{ds} = \frac{aQ - bP}{H_1 \sqrt{\Delta_1 \phi}}$$

If now in
$$\frac{da}{ds} = \frac{\partial a}{\partial u} \frac{du}{ds} + \frac{\partial a}{\partial v} \frac{dv}{ds}$$

we replace the expressions for $\frac{\partial a}{\partial u}$ and $\frac{\partial a}{\partial v}$ from (47), and similarly for db/ds and dc/ds , we obtain

$$\begin{aligned} Pds &= H_1 \sqrt{\Delta_1} \phi (p du + p_1 dv), & Qds &= H_1 \sqrt{\Delta_1} \phi (q du + q_1 dv), \\ Rds &= H_1 \sqrt{\Delta_1} \phi (r du + r_1 dv) \end{aligned}$$

From these equations and (79) we derive the following fundamental formulas

$$(80) \quad \left\{ \begin{aligned} \left(\frac{d\bar{\omega}}{ds} - \frac{1}{\tau} \right) ds &= \cos \Phi (p du + p_1 dv) + \sin \Phi (q du + q_1 dv), \\ \frac{\cos \bar{\omega}}{\rho} ds &= \sin \Phi (p du + p_1 dv) - \cos \Phi (q du + q_1 dv), \\ \frac{\sin \bar{\omega}}{\rho} &= \frac{d\Phi}{ds} + r \frac{du}{ds} + r_1 \frac{dv}{ds} \end{aligned} \right.$$

By means of the last of equations (80) we shall express the geodesic curvature of a curve in terms of the functions E, F, G , of their derivatives, and of the angle θ which the curve makes with the curve $v = \text{const}$. If we take the x -axis of the trihedral tangent to the curve $v = \text{const}$, we obtain from the last of (80), in consequence of (45),

$$\frac{1}{\rho_g} = \frac{d\theta}{ds} + \frac{\sqrt{E}}{\rho_{gu}} \frac{du}{ds} + \left(\frac{\sqrt{G}}{\rho_{gv}} - \frac{\partial \omega}{\partial v} \right) \frac{dv}{ds}$$

From (III, 15, 16) we obtain

$$\frac{\partial \omega}{\partial v} = \frac{1}{H} \left[\frac{F}{2EG} \left(G \frac{\partial E}{\partial v} + E \frac{\partial G}{\partial v} \right) - \frac{\partial F}{\partial v} \right]$$

When this value and the expressions for ρ_{gu} and ρ_{gv} (IV, 57) are substituted in the above equation, we have the formula desired

$$(81) \quad \frac{1}{\rho_g} = \frac{d\theta}{ds} + \frac{1}{H} \left(\frac{\partial F}{\partial u} - \frac{F}{2E} \frac{\partial E}{\partial u} - \frac{1}{2} \frac{\partial E}{\partial v} \right) \frac{du}{ds} + \frac{1}{2H} \left(\frac{\partial G}{\partial u} - \frac{F}{E} \frac{\partial E}{\partial v} \right) \frac{dv}{ds}$$

EXAMPLES

1 A necessary and sufficient condition that the origin of the trihedral T be the only point in the moving xy plane which generates a surface to which this plane is tangent, is that the surface be nondevelopable

Examine ρ so that the point of coordinates $(\rho, 0, 0)$ with respect to T shall be the point of coordinates $(0, 0, 0)$ with respect to T_0 . Examine the case when the parametric curves are minimal and the x -axis is tangent to the curve $v = \text{const.}$

When the parametric curves are minimal lines for both the surface and the surface T , it is necessary that

$$\begin{aligned} \eta &= i\xi, & \eta_1 &= -i\xi_1, & q &= -p, & q_1 &= p_1, \\ \eta &= -i\xi, & \eta_1 &= i\xi_1, & q &= p, & q_1 &= -p_1, \end{aligned}$$

so the parametric curves on the surface form a conjugate system, and the surface is minimal (cf § 55)

When the asymptotic lines on a surface form an orthogonal system, we

$$p\xi_1 + q\eta_1 = 0, \quad p_1\xi + q_1\eta = 0,$$

and the surface is minimal

When the lines of curvature are parametric, and the x -axis of T is tangent to the curve $v = \text{const.}$, equations (80) reduce to

$$\begin{aligned} \frac{1}{\tau} - \frac{d\bar{\omega}}{ds} &= \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \sin \Phi \cos \Phi, & \frac{\cos \bar{\omega}}{\rho} &= \frac{\cos^2 \Phi}{\rho_1} + \frac{\sin^2 \Phi}{\rho_2}, \\ \frac{\sin \bar{\omega}}{\rho} &= \frac{d\Phi}{ds} + \frac{1}{\rho_2 - \rho_1} \left(\frac{q}{p_1} \frac{\partial \rho_1}{\partial v} \frac{du}{ds} + \frac{p_1}{q} \frac{\partial \rho_2}{\partial u} \frac{dv}{ds} \right) \end{aligned}$$

When the second equation in Ex 5 is differentiated with respect to s , the equation is reducible to

$$\begin{aligned} \frac{s}{\rho} \frac{d\bar{\omega}}{ds} + \frac{\sin \bar{\omega}}{\rho} \left(3 \frac{d\bar{\omega}}{ds} - \frac{2}{\tau} \right) &= q^2 \frac{\partial \rho_1}{\partial u} \left(\frac{du}{ds} \right)^3 + 3 q^2 \frac{\partial \rho_1}{\partial v} \left(\frac{du}{ds} \right)^2 \frac{dv}{ds} \\ &\quad + 3 p_1^2 \frac{\partial \rho_2}{\partial u} \frac{du}{ds} \left(\frac{dv}{ds} \right)^2 + p_1^2 \frac{\partial \rho_2}{\partial v} \left(\frac{dv}{ds} \right)^3 \end{aligned}$$

When a surface is given, a given curve makes the angle Φ with the x axis of a trihedral T , P_0 of coordinates $\cos \Phi, \sin \Phi, 0$ with reference to the parallel trihedral T_0 . The vertex, describes the spherical indicatrix of the tangent to the curve, and the direction-cosines of the tangent to this curve are

$$-\sin \Phi \sin \bar{\omega}, \quad \cos \Phi \sin \bar{\omega}, \quad \cos \bar{\omega},$$

as the significance indicated in § 49, and the linear element is ds/ρ , derive by means of (51) the second and third of formulas (80)

At point B , whose coordinates with reference to T_0 of Ex 7 are

$$\sin \Phi \cos \bar{\omega}, \quad -\cos \Phi \cos \bar{\omega}, \quad \sin \bar{\omega},$$

the spherical indicatrix of the binormal to the given curve on the surface, near element is ds/τ , derive therefrom the first of formulas (80)

Parallel surfaces We inquire under what conditions the lines normal to a surface are normal to a second surface. In order that this be possible, there must exist a function t such that the point $P(t)$ of coordinates $(0, 0, t)$, with reference to the trihedral T , describes a curve to which the moving z -axis is constantly normal. Hence

we must have $\delta z = 0$, and consequently, by equations (51), t must be a constant, which may have any value whatever. We have, therefore, the theorem

If segments of constant length be laid off upon the normals to a surface, these segments being measured from the surface, the locus of their other end points is a surface with the same normals as the given surface

These two surfaces are said to be *parallel*. Evidently there is an infinity of surfaces parallel to a given surface, and all of them have the same spherical representation.

Consider the surface for which t has the value a , and call it $\bar{S}$. From (51) it follows that the projections on the axes of T of a displacement on $\bar{S}$ have the values

$$(82) \quad \begin{cases} \delta \bar{x} = \xi du + \xi_1 dv + (q du + q_1 dv) a, \\ \delta \bar{y} = \eta du + \eta_1 dv - (p du + p_1 dv) a \end{cases}$$

Comparing these results with (53), we see that the displacements on the two surfaces corresponding to the same value of dv/du are parallel only in case equation (59) is satisfied, that is, when the point describes a line of curvature on S . But from a characteristic property of lines of curvature (§ 51) it follows that the lines of curvature on the two surfaces correspond. Hence we have the theorem

The tangents to corresponding lines of curvature of two parallel surfaces at corresponding points are parallel

From (82) and (73) we have the following expressions for the first fundamental quantities of S

$$(83) \quad \begin{cases} E = E - 2aD + a\omega, \\ F = F - 2aD' + a'\omega', \\ G = G - 2aD'' + a\omega'', \end{cases}$$

or, in consequence of (IV, 78),

$$(84) \quad \begin{cases} E = E \left(1 - \frac{a}{\rho_1 \rho} \right) + Da \left[a \left(\frac{1}{\rho_1} + \frac{1}{\rho} \right) - 2 \right], \\ F = F \left(1 - \frac{a}{\rho_1 \rho} \right) + D'a \left[a \left(\frac{1}{\rho_1} + \frac{1}{\rho} \right) - 2 \right], \\ G = G \left(1 - \frac{a}{\rho_1 \rho} \right) + D''a \left[a \left(\frac{1}{\rho_1} + \frac{1}{\rho} \right) - 2 \right] \end{cases}$$

moving trihedral for $\bar{S}$ can be taken parallel to T for S , as the rotations are the same for both trihedrals, and from follows that the translations have the values

$$\xi + aq, \quad \bar{\xi}_1 = \xi_1 + aq_1, \quad \bar{\eta} = \eta - ap, \quad \bar{\eta}_1 = \eta_1 - ap_1$$

stituting in the equations for $\bar{S}$ analogous to (59), (60), (66), in the fundamental equations for $\bar{S}$ in terms of the functions Also from (73) we have the following expressions for the fundamental coefficients for $\bar{S}$

$$\bar{D} = D - a\mathcal{E}, \quad \bar{D}' = D' - a\mathcal{F}, \quad \bar{D}'' = D'' - a\mathcal{G}$$

the centers of principal curvature of a surface and its at corresponding points are the same, it follows that

$$\bar{\rho}_1 = \rho_1 + a, \quad \bar{\rho}_2 = \rho_2 + a$$

ose that we have a surface whose total curvature is constant al to $1/c^2$. Evidently a sphere of radius c is of this kind, r (Chapter VIII) it will be shown that there is a large group ces with this property. We call them *spherical surfaces*

$$(86) \text{ we have } (\bar{\rho}_1 - a)(\bar{\rho}_2 - a) = c^2,$$

if we take $a = \pm c$, we obtain

$$\frac{1}{\bar{\rho}_1} + \frac{1}{\bar{\rho}_2} = \pm \frac{1}{c}$$

we have the theorem of Bonnet *

every surface of constant total curvature $1/c^2$ there are asso-vo surfaces of mean curvature $\pm 1/c$, they are parallel to the and at the distances $\mp c$ from it

versely,

every surface whose mean curvature is constant and different o there are associated two parallel surfaces, one of which has total curvature and the other constant mean curvature

urfaces of center. As a point M moves over a surface S esponding centers of principal curvature M_1 and M describe faces S_1 and S , which ue called the *surfaces of center* of S nd C' be the lines of curvature of S through M , and D_1 and levelopable surfaces formed by the normals to S along C_1

and C_2 respectively. The edge of regression of D_1 , denoted by Γ_1 , is a curve on S_1 (see fig 17), and consequently S_1 is the locus of one set of evolutes of the curves C_1 on S . Similarly S_2 is the locus of a set of evolutes of the curves C_2 on S . For this reason S_1 and S_2 are said to constitute the *evolute* of S , and S is their *involute*. Evidently any surface parallel to S is also an involute of S_1 and S_2 .

The line M_1M_2 , as a generator of D_1 , is tangent to Γ_1 at M_1 , and, as a generator of D_2 , it is tangent to Γ_2 at M_2 . Hence it is a common tangent of the surfaces S_1 and S_2 . From this it follows that the developable surface D_1 meets S_1 along Γ_1 and envelopes S_2 along a curve Γ'_2 . Its generators are consequently tangent to the curves conjugate to Γ'_2 (§ 54). In particular, the generator M_1M_2 is tangent to Γ_2 , and therefore the directions of Γ_2 and Γ'_2 at M_2 are conjugate. Similar results follow from the consideration of D_2 . Hence

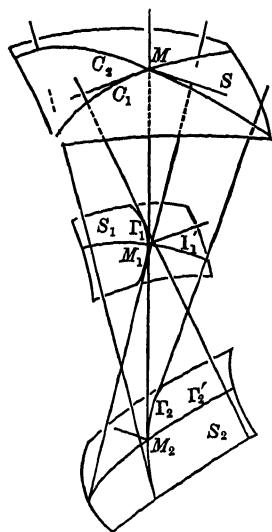


FIG 17

On the surfaces of center of a surface S the curves corresponding to the lines of curvature of S form a conjugate system

Since the developable D_1 envelopes S , the tangent plane to S_2 at M_2 is the tangent plane to D_1 at this point. But the tangent plane at M_2 is tangent to D_1 all along M_1M (§ 25), and consequently it is determined by M_1M and the tangent to C_1 at M . Hence the normal to S at M is parallel to the tangent to C_1 at M . In like manner, the normal to S_1 at M_1 is parallel to the tangent to C_2 at M . Thus, through each normal to S we have two perpendicular planes, of which one is tangent to one surface of center and the other to the second surface. But each of these planes is at the same time tangent to one of the developables, and is the osculating plane of its edge of regression. Hence, at every point of one of these curves, the plane is perpendicular to the tangent plane to the sheet of the evolute upon which it lies, and so we have the theorem

The edges of regression of the developable surfaces formed by the normals to a surface along the lines of curvature of one family are

on the surface of center which is the locus of these edges, developable surfaces formed by the normals along the lines of e in the other family envelope this surface of center along the conjugate to these geodesics

In the following sections we shall obtain, in an analytical manner, its just deduced geometrically

Fundamental quantities for surfaces of center As the trihedron moves over the surface S the point $(0, 0, \rho_1)$ describes the surface of center S_1 . Let the lines of curvature on S be parametric, x -axis of T be tangent, to the curve $v = \text{const}$. Now

$$\eta = \xi_1 = p = q_1 = 0, \quad q = -\frac{\sqrt{E}}{\rho_1}, \quad p_1 = \frac{\sqrt{G}}{\rho_2},$$

the first two of equations (48) may be put in the form

$$\begin{cases} \frac{\sqrt{E}}{\sqrt{G}} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) r_1 = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial u} = \frac{\partial}{\partial u} \left(\frac{1}{\rho_2} \right), \\ \frac{\sqrt{G}}{\sqrt{E}} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) r = \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \frac{1}{\sqrt{E}} \frac{\partial \sqrt{E}}{\partial v} = \frac{\partial}{\partial v} \left(\frac{1}{\rho_1} \right) \end{cases}$$

projections on the moving axes of the absolute displacement M_1 corresponding to a displacement of M on S are found 1) to be

$$\delta x_1 = 0, \quad \delta y_1 = (\eta_1 - \rho_1 p_1) dv = \sqrt{G} \left(1 - \frac{\rho_1}{\rho_2} \right) dv, \quad \delta z_1 = d\rho_1$$

the linear element of S_1 is

$$ds_1^2 = d\rho_1^2 + G \left(1 - \frac{\rho_1}{\rho_2} \right)^2 dv^2,$$

hence the curves $\rho_1 = \text{const}$ on S_1 are the orthogonal trajectories of the curves $v = \text{const}$, which are the edges of regression, the developables of the normals to S along the lines of e are $v = \text{const}$

we now consider the moving trihedral T_1 for S_1 formed by the tangents to the curves $v = \text{const}$ and $\rho_1 = \text{const}$ at M_1 and the normal at this point. From (89) it follows that the first tangent has the same direction and sense as the normal to S , and that the second tangent has the same direction as the tangent to $u = \text{const}$ on S , the sense being the same or different according as $(1 - \rho_1/\rho_2)$ is

positive or negative. And the normal to S_1 has the same direction as the tangent to $v = \text{const}$ on S , and the contrary or same sense accordingly.

If then we indicate with an accent quantities referring to the moving trihedral T_1 , we have

$$(91) \quad \begin{cases} x' = z - \rho_1, & y' = \epsilon y, & z' = -\epsilon x, \\ a' = c, & b' = \epsilon b, & c' = -\epsilon a, \end{cases}$$

where ϵ is ± 1 according as $(1 - \rho_1/\rho_2)$ is positive or negative. From (89) it follows that

$$(92) \quad \begin{cases} \xi' = \frac{\partial \rho_1}{\partial u}, & \xi'_1 = \frac{\partial \rho_1}{\partial v}, & \eta' = 0, & \eta'_1 = \epsilon \sqrt{G} \left(1 - \frac{\rho_1}{\rho_2}\right), \\ & & \zeta' = \zeta'_1 = 0 \end{cases}$$

When the values (91) are substituted in equations for T_1 similar to equations (47), we find

$$(93) \quad \begin{cases} p' = r, & q' = \epsilon q, & r' = -\epsilon p = 0, \\ p'_1 = r_1, & q'_1 = \epsilon q_1 = 0, & r'_1 = -\epsilon p_1 \end{cases}$$

Since r' is zero, it follows from (76) that the curves $v = \text{const}$ are geodesics, as found geometrically.

The various fundamental equations for S_1 may now be obtained by substituting these values in the corresponding equations of the preceding sections. Thus, from (73) we have

$$(94) \quad H_1 = \left(\frac{\partial \rho_1}{\partial u}\right)^2, \quad F_1 = \frac{\partial \rho_1}{\partial u} \frac{\partial \rho_1}{\partial v}, \quad G_1 = \left(\frac{\partial \rho_1}{\partial v}\right)^2 + G \left(1 - \frac{\rho_1}{\rho_2}\right)^2,$$

which follow likewise from (90), and also

$$(95) \quad D_1 = -\frac{\epsilon \sqrt{h}}{\rho_1} \frac{\partial \rho_1}{\partial u}, \quad D'_1 = 0, \quad D''_1 = -\frac{\epsilon \rho_1}{\rho_2^2} \frac{G}{\sqrt{h}} \frac{\partial \rho}{\partial u}$$

Hence the parametric curves on S_1 form a conjugate system (cf. § 54).

The equation of the lines of curvature may be written

$$(96) \quad r \frac{\partial \rho_1}{\partial u} du^2 + \left[r_1 \frac{\partial \rho_1}{\partial u} + r \frac{\partial \rho_1}{\partial v} + q p_1 (\rho - \rho_1) \right] du dv + r_1 \frac{\partial \rho_1}{\partial v} dv^2 = 0,$$

and the equation of the asymptotic directions is

$$(97) \quad \frac{h}{\rho_1^2} \frac{\partial \rho_1}{\partial u} du^2 - \frac{G}{\rho_2^2} \frac{\partial \rho}{\partial u} dv^2 = 0$$

expression for K_1 , the total curvature of S_1 , is

$$K_1 = - \frac{1}{(\rho_1 - \rho_2)^2} \frac{\frac{\partial \rho_2}{\partial u}}{\frac{\partial \rho_1}{\partial u}}$$

(80) and (93) it follows that the geodesic curvature at M_1 of e on S_1 which makes the angle Φ_1 with the curve $v = \text{const}$ M_1 is given by

$$\frac{1}{\rho_g} = \frac{d\Phi_1}{ds_1} - \epsilon p_1 \frac{dv}{ds_1}$$

the radius of geodesic curvature of a curve $\rho_1 = \text{const}$, that curve for which Φ_1 is a right angle, has, in consequence of a value $\rho_1 - \rho_2$. In accordance with § 57 the center of geodesic curvature is found by measuring off the distance $\rho_1 - \rho_2$, in the v direction, on the z -axis of the trihedral T . Consequently M_1 is center of curvature. Hence we have the following theorem:

Centers of geodesic curvature of the curves $\rho_1 = \text{const}$ on S_1 , $\rho_2 = \text{const}$ on S_2 are the corresponding points on S_2 and S_1 respectively.

On the sheet S_2 of the evolute we find the following results

$$ds_2^2 = E \left(1 - \frac{\rho_2}{\rho_1} \right)^2 du^2 + d\rho^2,$$

equation of the lines of curvature is

$$\frac{\partial \rho}{\partial u} du^2 + \left[r \frac{\partial \rho}{\partial v} + r_1 \frac{\partial \rho_2}{\partial u} + p_1 q (\rho - \rho_1) \right] du dv + r_1 \frac{\partial \rho}{\partial v} dv^2 = 0,$$

equation of the asymptotic lines is

$$\frac{r}{\rho_1^2} \frac{\partial \rho_1}{\partial v} du - \frac{r}{\rho_1} \frac{\partial \rho}{\partial v} dv = 0,$$

expression for the total curvature is

$$K = - \frac{1}{(\rho_1 - \rho_2)^2} \frac{\frac{\partial \rho_1}{\partial v}}{\frac{\partial \rho_2}{\partial v}}$$

In consequence of these results we are led to the following theorems of Ribaucour,* the proof of which we leave to the reader

A necessary and sufficient condition that the lines of curvature upon S_1 and S_2 correspond is that $\rho_1 - \rho_2 = c$ (a constant), then $K_1 = K_2 = -1/c^2$, and the asymptotic lines upon S_1 and S_2 correspond

A necessary and sufficient condition that the asymptotic lines on S_1 and S_2 correspond is that there exist a functional relation between ρ_1 and ρ_2

76 Surfaces complementary to a given surface We have just seen that the normals to a surface are tangent to a family of geodesics on each surface of centers. Now we prove the converse

The tangents to a family of geodesics on a surface S_1 are normal to an infinity of parallel surfaces

Let the geodesics and their orthogonal trajectories be taken for the curves $v = \text{const}$ and $u = \text{const}$ respectively, and the parameters chosen so that the linear element has the form

$$ds_1^2 = du^2 + G_1 dv^2$$

We refer the surface to the trihedral formed by the tangents to the parametric curves and the normal, the x -axis being tangent to the curve $v = \text{const}$. Upon the latter we lay off from the point M_1 of the surface a length λ , and let P denote the other extremity. As M_1 moves over the surface the projections of the corresponding displacements of P have the values

$$(99) \quad d\lambda + du, \quad \left(\sqrt{G_1} + \lambda \frac{\partial \sqrt{G_1}}{\partial u} \right) dv, \quad -\lambda (q du + q_1 dv)$$

In order that the locus of P be normal to the lines M_1P , we must have $d\lambda + du = 0$, and consequently

$$\lambda = -u + c,$$

where c denotes the constant of integration whose value determines a particular one of the family of parallel surfaces. If the direction-cosines of M_1P with reference to fixed axes be A_1, Y_1, Z_1 , the coordinates of the surface S , for which $c = 0$, are given by

$$x = x_1 - u\lambda_1, \quad y = y_1 - uY_1, \quad z = z_1 - uZ_1,$$

where x_1, y_1, z_1 are the coordinates of M_1

surface S_1 is one of the surfaces of center of S . In order the other, S_2 , we must determine λ so that the locus of P nt at P to the xz -plane of the moving trihedral. The con- or this is

$$\frac{\partial \sqrt{G_1}}{\partial u} \lambda + \sqrt{G_1} = 0$$

S_2 is given by

$$x_2 = x_1 - \frac{\sqrt{G_1}}{\frac{\partial \sqrt{G_1}}{\partial u}} X_1, \quad y_2 = y_1 - \frac{\sqrt{G_1}}{\frac{\partial \sqrt{G_1}}{\partial u}} Y_1, \quad z_2 = z_1 - \frac{\sqrt{G_1}}{\frac{\partial \sqrt{G_1}}{\partial u}} Z_1,$$

principal radii of S are expressed by

$$\rho_1 = u, \quad \rho_2 = u - \frac{\sqrt{G_1}}{\frac{\partial \sqrt{G_1}}{\partial u}}$$

* calls S_2 the *surface complementary* to S_1 for the given system

ami has suggested the following geometrical proof of the theorem. Of the involutes of the geodesics $v = \text{const}$ we the single infinity which meet S_1 in one of the orthogonal ries $u = u_0$. We shall prove that the locus of these curves face S , normal to the tangents to the geodesics. Consider ents to the geodesics at the points of meeting of the latter econd orthogonal trajectory $u = u_1$. The segments of these s between the points of contact M and the points P of with S are equal to one another, because they are equal ngth of the geodesics between the curves $u = u_0$ and $u = u_1$ as M moves along an orthogonal trajectory $u = u_1$ of the P , P describes a second orthogonal trajectory of the latter or, as M moves along a geodesic, P describes an involute s necessarily orthogonal to MP . Since two directions on S endicular to MP , the latter is normal to S .

EXAMPLES

ain the results of § 73 concerning parallel surfaces without making use of g trihedral

w that the surfaces parallel to a surface of revolution are surfaces of

3 Determine the conjugate systems upon a surface such that the corresponding curves on a parallel surface form a conjugate system

4 Determine the character of a surface S such that its asymptotic lines correspond to conjugate lines upon a parallel surface, and find the latter surface

5 Show that when the parametric curves are the lines of curvature of a surface, the characteristics of the yz -plane and xz plane respectively of the moving trihedral whose x -axis is tangent to the curve $v = \text{const}$ at the point are given by

$$(r du + r_1 dv)y - q(z - \rho_1) du = 0,$$

$$(r du + r_1 dv)x - p_1(z - \rho_2) dv = 0,$$

and show that these equations give the directions on the surfaces S_1 and S_2 which are conjugate to the direction determined by dv/du

6 Show that for a canal surface (§ 29) one surface of centers is the curve of centers of the spheres and the other is the polar developable of this curve

7 The surfaces of center of a helicoid are helicoids of the same axis and parameter as the given surface

GENERAL EXAMPLES

1 If t is an integrating factor of $\sqrt{F} du + \frac{F + iH}{\sqrt{E}} dv$, and t_0 the conjugate imaginary function, then $\Delta_2 \log \sqrt{t_0}$ is equal to the total curvature of the quadratic form $E du^2 + 2F dudv + G dv^2$, all the functions in the latter being real

2 Show that the sphere is the only real surface such that its first and second fundamental quadratic forms can be the second and first forms respectively of another surface

3 Show that there exists a surface referred to its lines of curvature with the linear element $ds^2 = e^{\alpha} (du^2 + dv^2)$, where α is a constant, and that the surface is developable

4 When a minimal surface is referred to its minimal lines

$$D = \phi(u) \quad D = 0, \quad D = \psi(v)$$

hence the lines of curvature and asymptotic lines can be found by quadratures

5 Establish the following identities in which the differential parameters are formed with respect to the linear element

$$\Delta_1(x, X) = -\frac{X_1}{\rho_1} \frac{1}{\rho} \quad \Theta(x, X) = X_1 X \begin{pmatrix} 1 & 1 \\ \rho_1 & \rho \end{pmatrix}$$

$$\Delta_1(x, Y) = -\frac{Y_1 Y_1}{\rho_1} \frac{1}{\rho} \quad \Theta(x, Y) = \frac{Y_1 Y_1}{\rho_1} \frac{1}{\rho}$$

6 Prove that (cf Ex 2, p 166)

$$\Delta_0 X = -\frac{X_1}{\sqrt{E}} \frac{\partial}{\partial u} \left(\frac{1}{\rho_1} + \frac{1}{\rho} \right) - \frac{X}{\sqrt{G}} \frac{\partial}{\partial v} \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right) - X \left(\frac{1}{\rho_1^2} + \frac{1}{\rho} \right)$$

$$\Delta_2 W = -\left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right) - W \left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} \right) - \frac{W_1}{\sqrt{E}} \frac{\partial}{\partial u} \left(\frac{1}{\rho_1} + \frac{1}{\rho} \right) - \frac{W_2}{\sqrt{G}} \frac{\partial}{\partial v} \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right)$$

ow that $x^2 + y^2 + z^2 = W^2 + \Delta_1^2 W$,

ntial parameter being formed with respect to (23)

necessary and sufficient condition that all the curves of an orthogonal a surface be geodesics is that the surface be developable

the geodesic curvature of the curves of an orthogonal system is constant from zero) all over the surface, the latter is a surface of constant negative

hen the linear element of a surface is in the form

$$ds^2 = du^2 + 2 \cos \omega \, du \, dv + dv^2,$$

etric curves are said to form an *equidistantal* system. Show that in this ördinates of the surface are integrals of the system

$$\frac{\frac{\partial^2 x}{\partial u \partial v}}{\frac{\partial y}{\partial u} \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} \frac{\partial y}{\partial v}} = \frac{\frac{\partial^2 y}{\partial u \partial v}}{\frac{\partial z}{\partial u} \frac{\partial x}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial z}{\partial u}} = \frac{\frac{\partial^2 z}{\partial u \partial v}}{\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial v} \frac{\partial x}{\partial u}}$$

the curves $v = \text{const}$, $u = \text{const}$ form an equidistantal system, the tan-
ie curves $v = \text{const}$ are orthogonal to the lines joining the centers of geo-
ature of the curves $u = \text{const}$ and of their orthogonal trajectories

all the displacements of a trihedral T corresponding to a small displac-
s vertex M over the surface there are two which reduce to rotations, they
n M describes either of the lines of curvature through the point, and the
tation are situated in the planes perpendicular to the lines of curvature,
passing through one of the centers of principal curvature

hen a surface is referred to its lines of curvature, the curves defined by

$$q^2 \frac{\partial \rho_1}{\partial u} du^2 + 3 q^2 \frac{\partial \rho_1}{\partial v} du^2 dv + 3 p_1^2 \frac{\partial \rho_2}{\partial u} du dv^2 + p_1^2 \frac{\partial \rho_2}{\partial v} dv^2 = 0$$

ie property that the normal sections in these directions at a point are
nes, or are superosculated by their circles of curvature (cf Ex 9, p 21,
177) These curves are called the *superosculating lines* of the surface

ow that the superosculating lines on a surface and on a parallel surface
d

ow that the Gauss equation (64) can be put in the following form due to

$$\frac{\sqrt{EG} \sin \omega}{\rho_1 \rho_2} = \frac{\partial}{\partial v} \left(\frac{\sqrt{E}}{\rho_{gv}} \right) - \frac{\partial}{\partial u} \left(\frac{\sqrt{G}}{\rho_{gu}} \right) + \frac{\partial^2 \omega}{\partial u \partial v}$$

and ρ_{gv} denote the radii of geodesic curvature of the curves $v = \text{const}$ and
respectively

hen the parametric curves form an orthogonal system, the equation of
ay be written

$$\frac{1}{\rho_1 \rho_2} = \frac{1}{\sqrt{G}} \frac{\partial}{\partial v} \left(\frac{1}{\rho_{gv}} \right) - \frac{1}{\sqrt{E}} \frac{\partial}{\partial u} \left(\frac{1}{\rho_{gu}} \right) - \frac{1}{\rho_{gu}^2} - \frac{1}{\rho_{gv}^2}$$

etermine the surfaces which are such that one of them and a parallel
monically the segment between the centers of principal curvature

18 Determine the surfaces which are such that one of them and a parallel admit of an equivalent representation (cf Ex 14, p 113) with lines of curvature corresponding

19 Derive the following properties of the surface

$$x = \frac{a^2 - b^2}{ab} \frac{uv}{u+v}, \quad y = \frac{\sqrt{a^2 - b^2}}{b} \frac{v\sqrt{b^2 - u^2}}{u+v}, \quad z = \frac{\sqrt{a^2 - b^2}}{a} \frac{u\sqrt{v^2 - a^2}}{u+v}$$

- (i) the parametric lines are plane lines of curvature,
- (ii) the principal radii of curvature are $\rho_1 = v$, $\rho_2 = -u$,
- (iii) the surface is algebraic of the fourth order,
- (iv) the surfaces of center are focal conics

20 Given a curve C upon a surface S and the ruled surface formed by the tangents to S which are perpendicular to C at its points M , the point of each generator MN at which the tangent plane to the ruled surface is perpendicular to the tangent plane at M to S is the center of geodesic curvature of C at M , when the ruled surface is developable, this center of geodesic curvature is the point of contact of MN with the edge of regression

21 If two surfaces have the same spherical representation of their lines of curvature, the locus of the point dividing the join of corresponding points in constant ratio is a surface with the same representation

22 The locus of the centers of geodesic curvature of a line of curvature is an evolute of the latter

23 Show that when E, F, G, D, D', D'' of a surface are functions of a single parameter, the surface is a helicoid, or a surface of revolution

CHAPTER VI

SYSTEMS OF CURVES GEODESICS

asymptotic lines We have said that the asymptotic lines on a surface are the double family of curves whose tangents at a point are determined in direction by the differential equation

$$D du^2 + 2 D' dudv + D'' dv^2 = 0$$

The directions are imaginary and distinct at an elliptic point, distinct at a hyperbolic point, and real and coincident at a parabolic point. If we exclude the latter points from our discussion, asymptotic lines may be taken for parametric curves. A necessary and sufficient condition that they be parametric is (§ 55)

$$D = D'' = 0$$

From (IV, 25) we have

$$K = -\frac{D'^2}{H^2} = -\frac{1}{\rho^2},$$

so as thus defined ρ is called the *radius of total curvature*. The dazzi equations (V, 13') may be written

$$\frac{\partial \log \rho}{\partial u} = 2 \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}, \quad \frac{\partial \log \rho}{\partial v} = 2 \begin{Bmatrix} 12 \\ 1 \end{Bmatrix},$$

and the condition of integrability is

$$\frac{\partial}{\partial u} \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} = \frac{\partial}{\partial v} \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}$$

or, in consequence of (V, 3) this is equivalent to

$$\frac{\partial}{\partial u} \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} = \frac{\partial}{\partial v} \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}$$

And we saw that K is a function of E, F, G and their derivative equations (3) are two conditions upon the coefficients of the quadratic form

$$E du^2 + 2 F dudv + G dv^2,$$

that it may be the linear element of a surface referred to its asymptotic lines. When these conditions are satisfied the function D' is given by (2) to within sign. Hence, if we make no distinction between a surface and its symmetric with respect to a point, from § 65 follows the theorem

A necessary and sufficient condition that a quadratic form (6) be the linear element of a surface referred to its asymptotic lines is that its coefficients satisfy equations (3), when they are satisfied, the surface is unique

For example, suppose that the total curvature of the surface is the same at every point, thus

$$K = -\frac{1}{a^2},$$

where a is a constant. In this case equations (3) are

$$G \frac{\partial E}{\partial v} - F \frac{\partial G}{\partial u} = 0, \quad F \frac{\partial E}{\partial v} - E \frac{\partial G}{\partial u} = 0,$$

which, since $H^2 \neq 0$, are equivalent to

$$\frac{\partial E}{\partial v} = 0, \quad \frac{\partial G}{\partial u} = 0$$

Hence E is a function of u alone, and G a function of v alone. By a suitable choice of the parameters these two functions may be given the value a^2 , so that the linear element of the surface can be written

$$(7) \quad ds^2 = a^2 (du^2 + 2 \cos \omega du dv + dv^2),$$

where ω denotes the angle between the asymptotic lines. Thus far the Codazzi equations are satisfied and only the Gauss equation (V, 12) remains to be considered. When the above values are substituted, this becomes

$$(8) \quad \frac{\partial^2 \omega}{\partial u \partial v} = \sin \omega$$

Hence to every solution of this equation there corresponds a surface of constant curvature $-\frac{1}{a^2}$ whose linear element is given by (7).

The equation of the lines of curvature is $du^2 - dv^2 = 0$, so that if we put $u + v = 2u_1$, $u - v = 2v_1$, the quantities u_1 and v_1 are parameters of the lines of curvature, and in terms of these the equation of the asymptotic lines is $du_1^2 - dv_1^2 = 0$. Hence, when either the asymptotic lines or the lines of curvature are known upon a surface of constant curvature, the other system can be found by quadratures.

When the asymptotic lines are parametric, the Gauss equations (V, 7) may be written

$$(9) \quad \begin{cases} \frac{\partial^2 \theta}{\partial u^2} + a \frac{\partial \theta}{\partial u} + b \frac{\partial \theta}{\partial v} = 0, \\ \frac{\partial^2 \theta}{\partial v^2} + a_1 \frac{\partial \theta}{\partial u} + b_1 \frac{\partial \theta}{\partial v} = 0, \end{cases}$$

, b , a_1 , b_1 are determinate functions of u and v , and in
ence of (5)

$$\frac{\partial a}{\partial v} = \frac{\partial b_1}{\partial u}$$

ely, if two such equations admit three real linearly inde-
integrals $f_1(u, v)$, $f_2(u, v)$, $f_3(u, v)$, the equations

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v)$$

surface on which the parametric curves are the asymptotic
or, by the elimination of a , b , a_1 , b_1 from the six equations
by replacing θ in (9) by x , y , z we get

$$\begin{vmatrix} \frac{\partial^2 x}{\partial u^2} & \frac{\partial^2 y}{\partial u^2} & \frac{\partial^2 z}{\partial u^2} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = 0, \quad \begin{vmatrix} \frac{\partial^2 x}{\partial v^2} & \frac{\partial^2 y}{\partial v^2} & \frac{\partial^2 z}{\partial v^2} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \end{vmatrix} = 0,$$

re equivalent to (1), in consequence of (IV, 2, 5) *

example, consider the equations

$$\frac{\partial^2 \theta}{\partial u^2} = 0, \quad \frac{\partial^2 \theta}{\partial v^2} = 0,$$

the general integral is $auv + bu + cv + d$, where a , b , c , d are constants
ng the fixed axes suitably, the most general form of the equations of the
ay be put in the form

$$x = b_1 u + c_1 v, \quad y = b_2 u + c_2 v, \quad z = a_3 uv + b_3 u + c_3 v$$

se equations it is seen that all the asymptotic lines are straight lines, so
urface is a quadric. Moreover, by the elimination of u and v from these
we have an equation of the form $z = ax^2 + 2hxy + by^2 + cx + dy$. Hence
e is a paraboloid.

Spherical representation of asymptotic lines From (IV, 77)

that the total curvature of a surface, referred to its asymp-
es, may be expressed in the form

$$K = -\frac{H'}{D'},$$

$\mathcal{F}^2 = \mathcal{E}\mathcal{G} - \mathcal{F}^2$, the linear element of the spherical represen-
eing $d\sigma^2 = \mathcal{E}du^2 + 2\mathcal{F}dudv + \mathcal{G}dv^2$

ux, Vol I p 138. It should be noticed that the above result shows that the
that equations (9) admit three independent integrals carries with it not only
l other conditions of integrability.

From this result and (2) it follows that*

$$(12) \quad \rho = \frac{D'}{\mathcal{H}}$$

Hence the fundamental relations (IV, 74) reduce to

$$(13) \quad E = \rho^2 \mathcal{E}, \quad F = -\rho^2 \mathcal{F}, \quad G = \rho^2 \mathcal{G},$$

and equations (V, 26) may be written

$$(14) \quad \frac{\partial x}{\partial u} = \frac{\rho}{\mathcal{H}} \left(\mathcal{F} \frac{\partial X}{\partial u} - \mathcal{E} \frac{\partial X}{\partial v} \right), \quad \frac{\partial x}{\partial v} = \frac{\rho}{\mathcal{H}} \left(-\mathcal{G} \frac{\partial X}{\partial u} + \mathcal{F} \frac{\partial X}{\partial v} \right)$$

Moreover, the Codazzi equations (V, 27) are reducible to

$$(15) \quad \frac{\partial \log \rho}{\partial u} = -2 \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}', \quad \frac{\partial \log \rho}{\partial v} = -2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}'$$

Consider now the converse problem

To determine the condition to be satisfied by a parametric system of lines on the sphere in order that they may serve as the spherical representation of the asymptotic lines on a surface

First of all, equations (15) must satisfy the condition of integrability. Then ρ is obtainable by a quadrature. The corresponding values of x, y, z found from equations (14) and from similar ones are the coordinates of a surface upon which the asymptotic lines are parametric. For, it follows from (14) that

$$(16) \quad \sum \frac{\partial x}{\partial u} \frac{\partial X}{\partial u} = 0, \quad \sum \frac{\partial x}{\partial v} \frac{\partial X}{\partial v} = 0$$

Furthermore, ρ is determined to within a constant factor, consequently the same is true of x, y, z , therefore the surface is unique to within homothetic transformations. Hence we have the following theorem of DINI

A necessary and sufficient condition that a double family of curves upon the sphere be the spherical representation of the asymptotic lines upon a surface is that $\mathcal{E}, \mathcal{F}, \mathcal{G}$ satisfy the equation

$$(17) \quad \frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' = \frac{\partial}{\partial v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}',$$

the corresponding surfaces are homothetic transforms of one another, and their Cartesian coordinates are found by quadratures

* The choice $\rho = -D/\mathcal{H}$ gives the surface symmetric to the one corresponding to (12), as is seen from (14), and hence may be neglected

then equations (1) obtain, the fundamental equations (V, 28) to the identities

$$\left\{ \begin{array}{ll} \left\{ \begin{array}{c} 11 \\ 1 \end{array} \right\} = \left\{ \begin{array}{c} 11 \\ 1 \end{array} \right\}' - 2 \left\{ \begin{array}{c} 12 \\ 2 \end{array} \right\}', & \left\{ \begin{array}{c} 22 \\ 2 \end{array} \right\} = \left\{ \begin{array}{c} 22 \\ 2 \end{array} \right\}' - 2 \left\{ \begin{array}{c} 12 \\ 1 \end{array} \right\}', \\ \left\{ \begin{array}{c} 12 \\ 1 \end{array} \right\} = - \left\{ \begin{array}{c} 12 \\ 1 \end{array} \right\}', & \left\{ \begin{array}{c} 12 \\ 2 \end{array} \right\} = - \left\{ \begin{array}{c} 12 \\ 2 \end{array} \right\}', \\ \left\{ \begin{array}{c} 11 \\ 2 \end{array} \right\} = - \left\{ \begin{array}{c} 11 \\ 2 \end{array} \right\}', & \left\{ \begin{array}{c} 22 \\ 1 \end{array} \right\} = - \left\{ \begin{array}{c} 22 \\ 1 \end{array} \right\}' \end{array} \right.$$

third and fourth of these equations are consequences also of (15),

Formulas of Lelievre **Tangential equations** In consequence of (V, 31) equations (14) may be put in the form

$$\frac{\partial x}{\partial u} = \epsilon \rho \left(Z \frac{\partial Y}{\partial u} - Y \frac{\partial Z}{\partial u} \right), \quad \frac{\partial x}{\partial v} = -\epsilon \rho \left(Z \frac{\partial Y}{\partial v} - Y \frac{\partial Z}{\partial v} \right),$$

where ϵ is ± 1 according as the curvature of the surface is positive or negative. Hence, if we put

$$\nu_1 = \sqrt{-\epsilon \rho} X, \quad \nu_2 = \sqrt{-\epsilon \rho} Y, \quad \nu_3 = \sqrt{-\epsilon \rho} Z,$$

we have the following formulas due to Lelievre *

$$\left\{ \begin{array}{ll} \frac{\partial x}{\partial u} = \begin{vmatrix} \nu_2 & \nu_3 \\ \frac{\partial \nu_2}{\partial u} & \frac{\partial \nu_3}{\partial u} \end{vmatrix}, & \frac{\partial x}{\partial v} = - \begin{vmatrix} \nu_2 & \nu_3 \\ \frac{\partial \nu_2}{\partial v} & \frac{\partial \nu_3}{\partial v} \end{vmatrix}, \\ \frac{\partial y}{\partial u} = \begin{vmatrix} \nu_3 & \nu_1 \\ \frac{\partial \nu_3}{\partial u} & \frac{\partial \nu_1}{\partial u} \end{vmatrix}, & \frac{\partial y}{\partial v} = - \begin{vmatrix} \nu_3 & \nu_1 \\ \frac{\partial \nu_3}{\partial v} & \frac{\partial \nu_1}{\partial v} \end{vmatrix}, \\ \frac{\partial z}{\partial u} = \begin{vmatrix} \nu_1 & \nu_2 \\ \frac{\partial \nu_1}{\partial u} & \frac{\partial \nu_2}{\partial u} \end{vmatrix}, & \frac{\partial z}{\partial v} = - \begin{vmatrix} \nu_1 & \nu_2 \\ \frac{\partial \nu_1}{\partial v} & \frac{\partial \nu_2}{\partial v} \end{vmatrix} \end{array} \right.$$

conditions of integrability of these equations are

$$\frac{\frac{\partial^2 \nu_1}{\partial u \partial v}}{\nu_1} = \frac{\frac{\partial^2 \nu_2}{\partial u \partial v}}{\nu_2} = \frac{\frac{\partial^2 \nu_3}{\partial u \partial v}}{\nu_3}$$

By means of (V, 22) and (15) we find from (20) that the common ratio of these equations is $\frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial u \partial v} - \mathcal{F}$. Consequently ν_1, ν_2, ν_3 are solutions of the equation

$$\frac{\partial^2 \theta}{\partial u \partial v} = \left(\frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial u \partial v} - \mathcal{F} \right) \theta$$

Conversely, we have the theorem

Given three particular integrals ν_1, ν_2, ν_3 of an equation of the form

$$(22) \quad \frac{\partial^2 \theta}{\partial u \partial v} = \lambda \theta,$$

where λ is any function whatever of u and v , the surface, whose coordinates are given by the corresponding quadratures (21), has the parametric curves for asymptotic lines, and the total curvature of the surface is measured by

$$(23) \quad K = -\frac{1}{\rho^2} = -\frac{1}{(\nu_1^2 + \nu_2^2 + \nu_3^2)^2}$$

For, from (21), it is readily seen that ν_1, ν_2, ν_3 are proportional to the direction-cosines of the normal to the surface. And if these direction cosines be given by (20), we are brought to (19), from which we see that the conditions (16) are satisfied.

Take, for example, the simplest case $\frac{\partial^2 \theta}{\partial u \partial v} = 0$, and three solutions

$$\nu_i = \phi_i(u) + \psi_i(v) \quad (i = 1, 2, 3)$$

The coordinates of the surface are

$$x = \psi_1 \phi_3 - \psi_3 \phi_1 + \int (\phi_1 \phi_3 - \phi_2 \phi_1) du - \int (\psi_1 \psi_3 - \psi_1 \psi_2) dv,$$

and similar expressions for y and z . When, in particular, we take

$$\phi_i(u) = a_i u + b_i, \quad \psi_i(v) = \alpha_i v + \beta_i,$$

the expressions for x, y, z are of the form $auv + bu + cv + d$ and consequently the surface is a paraboloid.

From equations (V, 22, 34) it follows that when the asymptotic lines are parametric, the tangential coordinates $\Lambda, \Gamma, \mathcal{L}, H$ are solutions of the equations

$$(24) \quad \begin{cases} \frac{\partial^2 \theta}{\partial u^2} = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}' \frac{\partial \theta}{\partial u} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' \frac{\partial \theta}{\partial v} - \mathcal{E} \theta, \\ \frac{\partial^2 \theta}{\partial v^2} = \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}' \frac{\partial \theta}{\partial u} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}' \frac{\partial \theta}{\partial v} - \mathcal{G} \theta \end{cases}$$

EXAMPLES

On a nondevelopable surface straight lines are the only plane asymptotic

asymptotic lines on a minimal surface form an orthogonal isothermal and their spherical images also form such a system

With that of all the surfaces with the linear element $ds^2 = du^2 + (u^2 + a^2) dv^2$, the parametric curves for asymptotic lines and another for lines of curvature determine these two surfaces

The normals to a ruled surface along a generator are parallel to a plane conversely, by means of the formulas of Lelievre, that if the normals to a surface along the asymptotic lines in one system are parallel to a plane, which contains the curve, the surface is ruled

If we take $v_1 = u$, $v_2 = v$, $v_3 = \phi(v)$, the formulas of Lelievre define the spherical right conoid

If the asymptotic lines in one system on a surface be represented on the great circles, the surface is ruled

Conjugate systems of parametric lines Inversions It is our

now to consider the case where the parametric lines of a surface form a conjugate system. As thus defined, the characteristic tangent plane, as it envelops the surface along a curve of the system, are the tangents to the curves $u = \text{const}$ at their points of intersection with the former curve, and similarly for a plane tangent along a curve $v = \text{const}$

The analytical condition that the parametric lines form a conjugate system is (§ 54)

$$D' = 0$$

It follows immediately from equations (V, 7) that x, y, z are solutions of an equation of the type

$$\frac{\partial^2 \theta}{\partial u \partial v} + a \frac{\partial \theta}{\partial u} + b \frac{\partial \theta}{\partial v} = 0,$$

where a and b are functions of u and v , or constants. By a method similar to that of § 77 we prove the converse theorem

If $(u, v), f_1(u, v), f_2(u, v)$ be three linearly independent real solutions of an equation of the type (26), the equations

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v)$$

represent a surface upon which the parametric curves form a conjugate

*

We have seen that the lines of curvature form the only orthogonal conjugate system. Hence, in order that the parametric lines on the surface (27) be lines of curvature, we must have

$$F = \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v} = 0$$

But this is equivalent to the condition that $x^2 + y^2 + z^2$ also be a solution of equation (26), as is seen by substitution. Hence we have the theorem of Darboux *

If $x, y, z, x^2 + y^2 + z^2$ are particular solutions of an equation of the form (26), the first three serve for the rectangular coordinates of a surface, upon which the parametric lines are the lines of curvature

Darboux † has applied this result to the proof of the following theorem

When a surface is transformed by an inversion into a second surface, the lines of curvature of the former become lines of curvature of the latter

By definition an inversion, or a transformation by reciprocal radii, is given by

$$(28) \quad x_1 = \frac{c^2 x}{x^2 + y^2 + z^2}, \quad y_1 = \frac{c^2 y}{x^2 + y^2 + z^2}, \quad z_1 = \frac{c^2 z}{x^2 + y^2 + z^2},$$

where c denotes a constant. From these equations we find that

$$(29) \quad (x^2 + y^2 + z^2)(x_1^2 + y_1^2 + z_1^2) = c^4,$$

and by solving for x, y, z ,

$$(30) \quad x = \frac{c^2 x_1}{x_1^2 + y_1^2 + z_1^2}, \quad y = \frac{c^2 y_1}{x_1^2 + y_1^2 + z_1^2}, \quad z = \frac{c^2 z_1}{x_1^2 + y_1^2 + z_1^2}$$

If, now, the substitution $\theta = \frac{\sigma}{x_1^2 + y_1^2 + z_1^2}$

be effected upon equation (26), the resulting equation in σ will admit, in consequence of (29) and (30), the solutions x_1, y_1, z_1, c^4 , and therefore is of the form

$$(31) \quad \frac{\partial^2 \sigma}{\partial u \partial v} + \alpha \frac{\partial \sigma}{\partial u} + \beta \frac{\partial \sigma}{\partial v} = 0$$

er, equation (26) admits unity for a particular solution, consequently $x_1^2 + y_1^2 + z_1^2$ is a solution of (31), which proves the theorem.

For example, we consider a cone of revolution. Its lines of curvature are the meridians of the cone and the circular sections. When a transformation by reciprocity, whose pole is any point, is applied to the cone, the transform S has two families of circles for its lines of curvature, in consequence of the above theorem. We note that circles and straight lines, not through the pole, are transformed into circles. Moreover, the cone is the envelope of a family of spheres whose centers lie on its axis, and also of the one-parameter family of tangent planes, the envelope of which passes through the vertex. Since tangency is preserved in this transformation, the transform S is in two ways the envelope of a family of spheres: all the spheres of one family pass through a point, and the centers of the spheres of the other family lie in the plane determined by the axis of the cone and the pole.

Surfaces of translation. The simplest form of equation (26) is

$$\frac{\partial^2 \theta}{\partial u \partial v} = 0,$$

In this case equations (27) are of the type

$$x = U_1 + V_1, \quad y = U_2 + V_2, \quad z = U_3 + V_3,$$

where U_1, U_2, U_3 are any functions whatever of u alone, and V_1, V_2, V_3 are any functions of v alone. This surface may be generated by translating a curve upon the curve

$$x_1 = U_1, \quad y_1 = U_2, \quad z_1 = U_3$$

in such a manner that each of its points describes a curve congruent with the curve

$$x_2 = V_1, \quad y_2 = V_2, \quad z_2 = V_3$$

in such a manner that each of its points describes a curve congruent with the first curve. For this reason the surface is called a *surface of translation*. From this method of generation, as also from equation (26), it follows that the tangents to the curves of one family are parallel to one another. Hence we have the theorem of Lie *

Developable enveloping a surface of translation along a generator is a cylinder

* *Math Annalen*, Vol XIV (1879) pp 332-337

Lie has observed that the surface defined by (32) is the locus of the mid-points of the joins of points on the curves

$$\begin{aligned}x_1 &= 2U_1, & y_1 &= 2U_2, & z_1 &= 2U_3, \\x_2 &= 2V_1, & y_2 &= 2V_2, & z_2 &= 2V_3\end{aligned}$$

It may be that these two sets of equations define the same curve in terms of different parameters. In this case the surface is the locus of the mid-points of all chords of the curve. These results are only a particular case of the following theorem, whose proof is immediate

The locus of the point which divides in constant ratio the joins of points on two curves, or all the chords of one curve, is a surface of translation, in the latter case the curve is an asymptotic line of the surface

When the equations of a surface of translation are of the form

$$x = U, \quad y = V, \quad z = U_1 + V_1,$$

the generators are plane curves whose planes are perpendicular. We leave it to the reader to show that in this case the asymptotic lines can be found by quadratures.

82 Isothermal-conjugate systems When the asymptotic lines upon a surface are parametric, the second quadratic form may be written $\lambda du dv$. When the surface is real, so also is this quadratic form. Therefore, according as the curvature of the surface is positive or negative, the parameters u and v are conjugate imaginary or real.

We consider the former case and put

$$u = u_1 + v_1, \quad v = u_1 - v_1,$$

when u_1 and v_1 are real. In terms of these parameters the second quadratic form is $\lambda (du_1 + dv_1)^2$. Hence the curves $u_1 = \text{const}$, $v_1 = \text{const}$ form a conjugate system, for which

$$(33) \quad D = D'', \quad D' = 0$$

Bianchi* has called a system of this sort *isothermal conjugate*. Evidently such a system bears to the second quadratic form an analytical relation similar to that of an isothermal-orthogonal system

rst quadratic form. In the latter case it was only necessary $-F^2$ be positive, and the analogous requirement, namely $\gamma^2 > 0$, is satisfied by surfaces of positive curvature. Hence theorems for isothermal-orthogonal systems (§§ 40, 41) are reduced into theorems concerning isothermal-conjugate systems. Substituting D, D', D'' for E, F, G respectively in the formulas above, we remark that if the curves $u = \text{const}$, $v = \text{const}$ on a surface form an isothermal-conjugate system, all other real isothermal-conjugate systems are given by $u_1 = \text{const}$, $v_1 = \text{const}$, the quantities u_1 and v_1 being defined by

$$u_1 + v_1 = \phi(u \pm v),$$

where ϕ is any analytic function.

If the curvature of the surface is negative and we put

$$u = u_1 + v_1, \quad v = u_1 - v_1,$$

the second quadratic form $\lambda du dv$, it becomes $\lambda(du_1^2 - dv_1^2)$. In the case

$$D = -D'', \quad D' = 0$$

the curves $u_1 = \text{const}$ and $v_1 = \text{const}$ form a conjugate system which may be called *isothermal-conjugate*. With each change of parameters u and v of the asymptotic lines there is obtained an isothermal-conjugate system. Hence if u and v are parameters of an isothermal-conjugate system upon a surface of negative curvature, the parameters of all such systems are given by

$$u_1 = \phi(u \pm v) + \psi(u \mp v),$$

$$v_1 = \phi(u \pm v) - \psi(u \mp v),$$

where ϕ and ψ denote arbitrary functions.

It is evident that if the parameters for a surface are such that

$$\frac{D}{D''} = \frac{U}{V}, \quad D' = 0,$$

where U and V are functions of u and v respectively, then by a change of parameters which does not change the parametric curves (35) reduce (35) to one of the forms (33) or (34). Hence equation (35) is a necessary and sufficient condition that the parametric curves form an isothermal conjugate system. Referring to

§ 77, we see that the lines of curvature upon a surface of constant total curvature form an isothermal-conjugate system

When equation (35) is of the form (33) or (34), we say that the parameters u and v are isothermal-conjugate

83 Spherical representation of conjugate systems When the parametric curves are conjugate, equations (IV, 69) reduce to

$$\mathcal{E} = \frac{GD^2}{H^2}, \quad \mathcal{F} = -\frac{FDD''}{H^2}, \quad \mathcal{G} = \frac{ED''^2}{H^2}$$

From these equations and (III, 15) it follows that the angle ω' between the parametric curves on the sphere is given by

$$\cos \omega' = \frac{\mathcal{F}}{\sqrt{\mathcal{E}\mathcal{G}}} = \mp \frac{F}{\sqrt{EG}} = \mp \cos \omega,$$

where the upper sign corresponds to the case of an elliptic point and the lower to a hyperbolic point. Hence we have the theorem

The angles between two conjugate directions at a point on a surface, and between the corresponding directions on the sphere, are equal or supplementary, according as the point is hyperbolic or elliptic

When the parametric curves form a conjugate system, the Codazzi equations (V, 27) reduce to

$$(36) \quad \begin{cases} \frac{\partial D}{\partial v} = \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}' D - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' D'', \\ \frac{\partial D''}{\partial u} = \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}' D'' - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}' D, \end{cases}$$

and equations (V, 26) become

$$(37) \quad \begin{cases} \frac{c_1}{cu} = \frac{D}{f} \left(-\zeta \frac{\partial \lambda}{cu} + \eta \frac{\partial \lambda}{cv} \right), \\ \frac{c_2}{cv} = \frac{D''}{f} \left(\eta \frac{\partial \lambda}{cu} - \zeta \frac{\partial \lambda}{cv} \right) \end{cases}$$

Hence, when a system of curves upon the sphere is given, the problem of finding the surfaces with this representation of a conjugate system reduces to the solution of equations (36) and quadratures of the form (37), after X , Y , Z have been determined by the solution of a Riccati equation. By the elimination of D or D'' from equations (36) we obtain a partial differential equation of the second order

the general equations (V, 28) we derive the following, the parametric curves form a conjugate system *

$$\begin{cases} \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} = \frac{\partial \log D}{\partial u} - \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}', & \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} = \frac{\partial \log D''}{\partial v} - \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}', \\ \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} = -\frac{D''}{D} \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}', & \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} = -\frac{D}{D''} \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}', \\ \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} = -\frac{D''}{D} \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}', & \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} = -\frac{D}{D''} \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}' \end{cases}$$

tangential coordinates **Projective transformations** The problem of finding the surfaces with a given representation of a conic system is treated more readily from the point of view of tangential coordinates. For, from (V, 22) and (V, 34) it is seen that X , Z , and W are particular solutions of the equation

$$\frac{\partial^2 \theta}{\partial u \partial v} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}' \frac{\partial \theta}{\partial u} - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}' \frac{\partial \theta}{\partial v} + \mathcal{F} \theta = 0$$

every solution of this equation linearly independent of X , Z , and W determines a surface with the given representation of a conic system, and the calculation of the coordinates x , y , z involves quadratures (§ 67)

Moreover, it is readily seen that if the tangential coordinates u , v satisfy an equation of the form

$$\frac{\partial^2 \theta}{\partial u \partial v} + a \frac{\partial \theta}{\partial u} + b \frac{\partial \theta}{\partial v} + c \theta = 0,$$

the lines $u = \text{const}$ and $v = \text{const}$ form a conjugate system on the surface

For example, we determine the surfaces whose lines of curvature are represented on the sphere by a family of curves of constant geodesic curvature and their orthogonal trajectories. If the former family be the curves $v = \text{const}$, and if the latter be written $d\sigma^2 = E du^2 + G dv^2$, we must have (IV, 50)

$$\frac{1}{\sqrt{EG}} \frac{\partial \sqrt{E}}{\partial v} = \phi(v),$$

where ϕ is a function of v alone. By a change of the parameter v this may be reduced to unity. In this case equation (39) is reducible to

$$\frac{\partial}{\partial u} \left(\frac{\partial \theta}{\partial v} - \sqrt{G} \theta \right) = \frac{\partial \log \sqrt{G}}{\partial u} \left(\frac{\partial \theta}{\partial v} - \sqrt{G} \theta \right)$$

The general integral of this equation is

$$\theta = e^{\int_{v_0}^v \sqrt{G} dv} \left[U + \int_{v_0}^v V \sqrt{G} e^{-\int_{v_0}^v \sqrt{G} dv} dv \right],$$

where v_0 denotes a constant value of v , and U and V are arbitrary functions of u and v respectively. Hence

The determination of all the surfaces whose lines of curvature are represented on the sphere by a family of curves of constant geodesic curvature and their orthogonal trajectories, requires two quadratures

In order that among all the surfaces with the same representation of a conjugate system there may be a surface for which the system is isothermal-conjugate, and the parameters be isothermal conjugate, it is necessary that equations (36) be satisfied by $D'' = \pm D$, according as the total curvature is positive or negative. In this case equations (36) are

$$\frac{\partial \log D}{\partial v} = \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \mp \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}', \quad \frac{\partial \log D}{\partial u} = \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \mp \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}'$$

The condition of integrability is

$$(40) \quad \frac{\partial}{\partial u} \left[\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \mp \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' \right] = \frac{\partial}{\partial v} \left[\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \mp \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' \right]$$

When this is satisfied D may be found by quadratures, and then the coordinates, by (37). Hence we have the theorem

A necessary and sufficient condition that a family of curves upon the sphere represent an isothermal conjugate system on a surface, and that u and v be isothermal-conjugate parameters, is that $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ satisfy (40), then the surface is unique to within its homothetics, and its coordinates are given by quadratures

The following theorem concerning the invariance of conjugate directions and asymptotic lines is due to Darboux.

When a surface is subjected to a projective transformation or a transformation by reciprocal polars, conjugate directions and asymptotic lines are preserved

We prove this theorem geometrically. Consider a curve C on a surface S and the developable D circumscribing the surface along C . When a projective transformation is effected upon S we obtain a surface S_1 , corresponding point with point to S , and C goes into a curve C_1 upon S_1 , and D into a developable D_1 circumscribing S_1 along

Moreover, the tangents to C and C_1 correspond, as do the generators D and D_1 . Since the generators are in each case tangent to lines conjugate to C and C_1 respectively, the theorem is proved. In the case of a polar reciprocal transformation a plane corresponds to a point and vice versa, in such a way that a plane and a point go into a point and a plane through it. Hence S goes to C , C into D_1 , D into C_1 , and the tangents to C and generators to the generators of D_1 and tangents to C_1 . Hence the theorem is proved.

EXAMPLES

Show that the parametric curves on the surface

$$x = \frac{U_1 + V_1}{U + V}, \quad y = \frac{U_2 + V_2}{U + V}, \quad z = \frac{U_3 + V_3}{U + V},$$

where U 's are functions of u alone and the V 's of v alone, form a conjugate system on the surface $x = U_1 V_1, y = U_2 V_1, z = V_3$, where U_1, U_2 are functions of u and V_1, V_3 of v alone, the parametric curves form a conjugate system and isotropic lines can be found by quadratures.

The generators of a surface of translation form an equidistant system (p. 187).

Show that a paraboloid is a surface of translation in more than one way.

The locus of the mid-points of the chords of a circular helix is a right helicoid. Discuss the surface of translation which is the locus of points dividing in a fixed ratio the chords of a twisted cubic.

From (28) it follows that

$$dx_1^2 + dy_1^2 + dz_1^2 = \frac{c^4(dx^2 + dy^2 + dz^2)}{(x^2 + y^2 + z^2)},$$

that the transformation by reciprocal radii is conformal.

Determine the condition to be satisfied by the function ω so that a surface element $ds^2 = a^2(\cos^2 \omega du^2 + \sin^2 \omega dv^2)$

has the total curvature $-1/a^2$. Show that if the parametric curves are the lines of curvature, they form an isothermal conjugate system.

Find the necessary and sufficient condition that the line element of a surface be of a conjugate system can be written

$$ds^2 = \rho^2(\mathcal{E} du^2 - 2\mathcal{F} du dv + \mathcal{G} dv^2),$$

where the parametric curves be the characteristic lines. Find the condition imposed on the curves on the unit sphere in order that they may represent these lines.

Conjugate systems and asymptotic lines are transformed into curves of the same type when a surface is transformed by the general projective transformation

$$x = \frac{A}{D}, \quad y = \frac{B}{D}, \quad z = \frac{C}{D},$$

where A, B, C, D are linear functions of the new coordinates x_1, y_1, z_1 .

85 Equations of geodesic lines We have defined a geodesic to be a curve whose geodesic curvature is zero at every point, consequently its osculating plane at any point is perpendicular to the tangent plane to the surface

From (IV, 49) it follows that every geodesic upon a surface is an integral curve of the differential equation

$$(41) \quad \left(E \frac{du}{ds} + F \frac{dv}{ds} \right) \left(F \frac{d^2u}{ds^2} + G \frac{d^2v}{ds^2} \right) - \left(F \frac{du}{ds} + G \frac{dv}{ds} \right) \left(E \frac{d^2u}{ds^2} + F \frac{d^2v}{ds^2} \right) \\ + \left(E \frac{du}{ds} + F \frac{dv}{ds} \right) \left[\left(\frac{\partial F}{\partial u} - \frac{1}{2} \frac{\partial E}{\partial v} \right) \left(\frac{du}{ds} \right)^2 + \frac{\partial G}{\partial u} \frac{du}{ds} \frac{dv}{ds} + \frac{1}{2} \frac{\partial G}{\partial v} \left(\frac{dv}{ds} \right)^2 \right] \\ - \left(F \frac{du}{ds} + G \frac{dv}{ds} \right) \left[\frac{1}{2} \frac{\partial E}{\partial u} \left(\frac{du}{ds} \right)^2 + \frac{\partial E}{\partial v} \frac{du}{ds} \frac{dv}{ds} + \left(\frac{\partial F}{\partial v} - \frac{1}{2} \frac{\partial G}{\partial u} \right) \left(\frac{dv}{ds} \right)^2 \right] = 0$$

If the fundamental identity

$$E \left(\frac{du}{ds} \right)^2 + 2F \frac{du}{ds} \frac{dv}{ds} + G \left(\frac{dv}{ds} \right)^2 = 1,$$

which gives the relation between u , v , s along the curve, be differentiated with respect to s , we have

$$2 \frac{du}{ds} \left(E \frac{d^2u}{ds^2} + F \frac{d^2v}{ds^2} \right) + 2 \frac{dv}{ds} \left(F \frac{d^2u}{ds^2} + G \frac{d^2v}{ds^2} \right) + \frac{\partial E}{\partial u} \left(\frac{du}{ds} \right)^3 \\ + \left(\frac{\partial E}{\partial v} + 2 \frac{\partial F}{\partial u} \right) \left(\frac{du}{ds} \right)^2 \frac{dv}{ds} + \left(2 \frac{\partial F}{\partial v} + \frac{\partial G}{\partial u} \right) \frac{du}{ds} \left(\frac{dv}{ds} \right)^2 + \frac{\partial G}{\partial v} \left(\frac{dv}{ds} \right)^3 = 0$$

If this equation and (41) be solved with respect to $\left(E \frac{d^2u}{ds^2} + F \frac{d^2v}{ds^2} \right)$ and $\left(F \frac{d^2u}{ds^2} + G \frac{d^2v}{ds^2} \right)$, we obtain

$$E \frac{d^2u}{ds^2} + F \frac{d^2v}{ds^2} + \frac{1}{2} \frac{\partial E}{\partial u} \left(\frac{du}{ds} \right)^2 + \frac{\partial F}{\partial v} \frac{du}{ds} \frac{dv}{ds} + \left(\frac{\partial F}{\partial v} - \frac{1}{2} \frac{\partial E}{\partial u} \right) \left(\frac{dv}{ds} \right)^2 = 0, \\ F \frac{d^2u}{ds^2} + G \frac{d^2v}{ds^2} + \left(\frac{\partial F}{\partial u} - \frac{1}{2} \frac{\partial G}{\partial v} \right) \left(\frac{du}{ds} \right)^2 + \frac{\partial G}{\partial u} \frac{du}{ds} \frac{dv}{ds} + \frac{1}{2} \frac{\partial G}{\partial v} \left(\frac{dv}{ds} \right)^2 = 0$$

If these equations be solved with respect to $\frac{d^2u}{ds^2}$ and $\frac{d^2v}{ds^2}$, we have, in consequence of (V, 2),

$$(42) \quad \begin{cases} \frac{d^2u}{ds^2} + \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} \left(\frac{du}{ds} \right)^2 + 2 \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \frac{du}{ds} \frac{dv}{ds} + \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} \left(\frac{dv}{ds} \right)^2 = 0, \\ \frac{d^2v}{ds^2} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \left(\frac{du}{ds} \right)^2 + 2 \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \frac{du}{ds} \frac{dv}{ds} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} \left(\frac{dv}{ds} \right)^2 = 0 \end{cases}$$

pair of solutions of these equations of the form $u = f_1(s)$,
 , determines a geodesic on the surface, and s is its arc
 , geodesic may be defined in terms of u and v alone, without
 reduction of the parameter s . If $v = \phi(u)$ defines such a
 then

$$\frac{dv}{ds} = \phi' \frac{du}{ds}, \quad \frac{d^2v}{ds^2} = \phi'' \left(\frac{du}{ds} \right)^2 + \phi' \frac{d^2u}{ds^2}$$

Putting these expressions in (42) and eliminating $\frac{d^2u}{ds^2}$, we
 , within the factor $(du/ds)^2$,

$$\begin{aligned} \phi'' - \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} \phi'^3 + \left(\left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} - 2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} \right) \phi'^2 \\ + \left(2 \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} - \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\} \right) \phi' + \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} = 0 \end{aligned}$$

(42) it follows that when du/ds is zero,

$$\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} = \frac{1}{2H^2} \left(2G \frac{\partial F}{\partial v} - F \frac{\partial G}{\partial v} - G \frac{\partial G}{\partial u} \right) = 0$$

when this condition is not satisfied, equation (43) defines
 geodesics on a surface, and when it is satisfied, equations (43)
 const define them

the theory of differential equations it follows that there
 unique integral of (43) which takes a given value for
 and whose first derivative takes a given value for $u = u_0$,
 we have the fundamental theorem

*Through every point on a surface there passes a unique geodesic
 in every direction*

example, we consider the geodesics on a surface of revolution. We have
 (6) that the linear element of such a surface referred to its meridians and
 s of the form

$$ds^2 = (1 + \phi'^2) du^2 + u^2 dv^2,$$

$\phi(u)$ is the equation of the meridian curve. If we put

$$u_1 = \int \sqrt{1 + \phi'^2} du,$$

the inverse of this equation by $u = \psi(u_1)$, we have

$$ds^2 = du_1^2 + \psi^2 dv^2,$$

meridians and parallels are still the parametric curves. For this case equa
 are

$$\frac{d^2u_1}{ds^2} - \psi\psi' \left(\frac{dv}{ds} \right)^2 = 0 \quad \frac{d^2v}{ds^2} + \frac{2\psi'}{\psi} \frac{du_1}{ds} \frac{dv}{ds} = 0$$

The first integral of the second is

$$\psi^2 \frac{dv}{ds} = c,$$

where c is a constant. Eliminating ds from this equation and (47), and integrating, we have

$$(49) \quad c \int \frac{du_1}{\psi \sqrt{\psi^2 - c^2}} = \pm v + c_1,$$

where c_1 is a constant. The meridians $v = \text{const}$ correspond to the case $c = 0$. Hence we have the theorem

The geodesics upon a surface of revolution referred to its meridians and parallels can be found by quadratures

It should be remarked that equation (49) defines the geodesics upon any surface applicable to a surface of revolution.

86 Geodesic parallels Geodesic parameters From (43) it follows that a necessary and sufficient condition that the curves $v = \text{const}$ on a surface be geodesics is that

$$(50) \quad \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} = \frac{1}{2H^2} \left(2E \frac{\partial F}{\partial u} - F \frac{\partial E}{\partial u} - E \frac{\partial E}{\partial v} \right) = 0$$

If the parametric system be orthogonal, this condition makes it necessary that E be a function of u alone, say $E = U^2$. By replacing $\int U du$ by u we do not change the parametric lines, and E becomes equal to unity. And the linear element has the form

$$(51) \quad ds^2 = du^2 + G dv^2,$$

where in general G is a function of both u and v . From this it follows that the length of the segment of a curve $v = \text{const}$ between the curves $u = u_0$ and $u = u_1$ is given by

$$\int_{u_0}^{u_1} ds_u = \int_{u_0}^{u_1} du = u_1 - u_0$$

Since this length is independent of v , it follows that the segments of all the geodesics $v = \text{const}$ included between any two orthogonal trajectories are of equal length. In consequence of the fundamental theorem, we have that there is a unique family of geodesics which are the orthogonal trajectories of a given curve C . The above results enable us to state the following theorem of Gauss[†]

If geodesics be drawn orthogonal to a curve C , and equal lengths be measured upon them from C , the locus of their ends is an orthogonal trajectory of the geodesics

[†] I C, p. 25

gives us a means of finding all the orthogonal trajectories of a family of geodesics, when one of them is known. And it suggests the name *geodesic parallels* for these trajectories. Referring to § 37, we see that these are the curves there called *parallels*, and the theorem of § 37 may be stated thus:

Necessary and sufficient condition that the curves $\phi = \text{const}$ be geodesic parallels is that

$$\Delta_1 \phi = f(\phi),$$

where the differential parameter is formed with respect to the linear element of the surface, and f denotes any function. In order that ϕ be the length of the geodesic curves measured from the curve $\phi = 0$, the necessary and sufficient condition is that

$$\Delta_1 \phi = 1$$

For, we have seen that when a function ϕ satisfies (52), a function satisfying (53) can be found by quadrature. When u is taken as u , the linear element has the form (51). In this case we shall call u and v *geodesic parameters*.

Geodesic polar coordinates. The following theorem, due to M. A. Schwarz,* suggests an important system of geodesic parameters.

Geodesic lengths be laid off from a point P on the geodesics through P , and the end points of these lengths are the end points of an orthogonal trajectory of the geodesics.

In proving the theorem we take the geodesics for the curves $u = \text{const}$, and let u denote distances measured along these geodesics from P . The points of a curve $u = \text{const}$ are consequently at the same *geodesic distance* from P , and so we call them *geodesic circles*.

It is our problem to show that this parametric system is orthogonal.

In the choice of u we know that $E = 1$, and hence from (50) we see that F is independent of u . At P , that is for $u = 0$, the partial derivatives $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial v}$, $\frac{\partial z}{\partial v}$ are zero. Consequently F and G are zero at P .

Since G is independent of u , and the former, being independent of u , is always zero, the theorem is proved.

Consider such a system and two points $M_0(u, 0)$, $M_1(u, v_1)$ on the same geodesic circle of radius u . The length of the arc M_0M_1

is given by $\int_0^{v_1} \sqrt{G} dv$. As u approaches zero the ratio $M_1 M_0 / u$ approaches the angle between the tangents at P to the geodesics $v = 0$ and $v = v_1$. If θ denotes this angle, we have

$$\theta = \lim_{u=0} \frac{\int_0^{v_1} \sqrt{G} dv}{u} = \int_0^{v_1} \left[\frac{\partial \sqrt{G}}{\partial u} \right]_{u=0} dv$$

In order that v be θ , it is necessary and sufficient that $\left[\frac{\partial \sqrt{G}}{\partial u} \right]_{u=0} = 1$

These particular geodesic coordinates are similar to polar coordinates in the plane, and for this reason are called *geodesic polar coordinates*. The above results may now be stated thus

The necessary and sufficient conditions that a system of geodesic coordinates be polar are

$$(54) \quad \left[\sqrt{G} \right]_{u=0} = 0, \quad \left[\frac{\partial \sqrt{G}}{\partial u} \right]_{u=0} = 1$$

It should be noticed, however, that it may be necessary to limit the part of the surface under consideration in order that there be a one-to-one correspondence between a point and a pair of coordinates. For, it may happen that two geodesics starting from P meet again, in which case the second point of meeting would be defined by two sets of coordinates*. For example, the helices are geodesics on a cylinder (§ 12), and it is evident that any number of them can be made to pass through two points at a finite distance from one another by varying the angle under which they cut the elements of the cylinder. Hence, in using a system of geodesic polar coordinates with pole at P , we consider the portion of the surface inclosed by a geodesic circle of radius r , where r is such that no two geodesics through P meet within the circle†.

When the linear element is in the form (51), the equation of Gauss (V, 12) reduces to

$$(55) \quad \frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial u^2} = -K$$

If K_0 denotes the total curvature of the surface at the pole P , which by hypothesis is not a parabolic point, from (54) and (55) it follows that

$$\left[\frac{\partial^2 \sqrt{G}}{\partial u^2} \right]_{u=0} = 0, \quad \left[\frac{\partial^3 \sqrt{G}}{\partial u^3} \right]_{u=0} = -K_0$$

* Notice that the pole is a singular point for such a system, because $H^2 = 0$ for $u = 0$.

† Darboux (Vol II, p 408) shows that such a function r exists, this is suggested also by § 94.

re, for sufficiently small values of u , we have

$$\sqrt{G} = u - \frac{K_0 u^3}{6} +$$

the circumference and area of a geodesic circle of radius u ; values *

$$C = \int_0^{2\pi} \sqrt{G} dv = 2\pi u - \frac{\pi K_0 u^3}{3} + \epsilon_1,$$

$$A = \int_0^u \int_0^{2\pi} \sqrt{G} dudv = \pi u^2 - \frac{\pi K_0 u^4}{12} + \epsilon_2,$$

and ϵ_1 and ϵ_2 denote terms of orders higher than the third and respectively

EXAMPLES

and the geodesics of an ellipsoid of revolution

the equations $x = u$, $y = v$ define a representation of a surface with the element $ds^2 = v(du^2 + dv^2)$ upon the xy -plane in such a way that geodesics mer are represented by parabolas on the latter

and the total curvature of a surface with the linear element

$$ds^2 = R^2 \frac{(a^2 - v^2) du^2 + 2uv dudv + (a^2 - u^2) dv^2}{(a^2 - u^2 - v^2)^2},$$

and a are constants and integrate the equation of geodesics for the surface twisted curve is a geodesic on its rectifying developable

the evolutes of a twisted curve are geodesics on its polar developable

along a geodesic on a surface of revolution the product of the radius of the through a point and the sine of the angle of inclination of the geodesic meridian is constant

on a surface of revolution a curve cannot be a geodesic and loxodromic at the same time unless the surface be cylindrical

on a helicoid the orthogonal trajectories of the helices are geodesics and geodesics can be found by quadratures

a family of geodesics and their orthogonal trajectories on a surface form a conformal system, the surface is applicable to a surface of revolution

the radius of curvature of a geodesic on a cone of revolution at a point P is the cube of the distance of P from the vertex

area of a geodesic triangle With the aid of geodesic polar coordinates Gauss proved the following important theorem †

Excess over 180° of the sum of the angles of a triangle formed by geodesics on a surface of positive curvature, or the deficit from 180°

and *Journal de Mathématiques*, Ser 1, Vol XIII (1848), pp 80-86 † *L c* p 30

of the sum of the angles of such a triangle on a surface of negative curvature, is measured by the area of the part of the sphere which represents that triangle

In the proof of this theorem Gauss made use of the equation of geodesic lines in the form

$$(56) \quad d\theta + \frac{1}{H} \left(\frac{\partial F}{\partial u} - \frac{F}{2E} \frac{\partial E}{\partial u} - \frac{1}{2} \frac{\partial E}{\partial v} \right) du + \frac{1}{2H} \left(\frac{\partial G}{\partial u} - \frac{F}{E} \frac{\partial E}{\partial v} \right) dv = 0,$$

where θ denotes the angle which the tangent to a geodesic at a point makes with the curve $v = \text{const}$ through the point. This equation is an immediate consequence of formula (V, 81). When the parametric system is polar geodesic, this becomes

$$(57) \quad d\theta = -\frac{\partial \sqrt{G}}{\partial u} dv$$

Let ABC be a triangle whose sides are geodesics, and let α, β, γ denote the included angles. From (IV, 73) it follows that the inclosed area on the sphere is given by

$$(58) \quad \mathcal{A} = \iint H' du dv = \epsilon \iint KH du dv,$$

where ϵ is ± 1 according as the curvature is positive or negative, and the double integrals are taken over the respective areas.

Let A be the pole of a polar geodesic system and AB the curve $v = 0$. From (55) and (58) we have

$$\mathcal{A} = -\epsilon \int_0^\alpha \int_0^u \frac{\partial^2 \sqrt{H'}}{\partial u \partial v} dv du$$

In consequence of (54) we have, upon integration with respect to u ,

$$\mathcal{A} = \epsilon \int_0^\alpha \left(1 - \frac{\partial \sqrt{H'}}{\partial u} \right) dv,$$

which, by (57), is equivalent to

$$\mathcal{A} = \epsilon \int_0^\alpha dv + \epsilon \int_\pi^\beta d\theta$$

For, at B the geodesic BC makes the angle $\pi - \beta$ with the curve $v = 0$, and at C it makes the angle γ with the curve $v = \alpha$. Hence we have

$$\mathcal{A} = \epsilon(\alpha + \beta + \gamma - \pi),$$

which proves the theorem.

use of the form of the second part of (58) $\mathcal{A}$ may be said to be the total curvature of the geodesic triangle, so that the theorem may also be stated thus

Total curvature of a geodesic triangle is equal to the excess or deficit from 180° , of the sum of the angles of the triangle according as the curvature is positive or negative

Extension of these theorems to the case of geodesic polygons is straightforward

In the preceding discussion it has been tacitly assumed that all the points of the arc BC can be uniquely defined by polar coordinates with pole at A . We assume that this theorem is true, even if this assumption is

not true for ABC , it cannot be true for the triangles ABD and ACD obtained by joining A to a middle point D of BC with a geodesic AD (fig 18). For, the results for the two triangles, we should have the same for ABC . Suppose that it is not true for ABD ; divide it into two triangles and apply the same reasoning; continuing this process we should obtain a triangle as we please, inside of which a polar geodesic system

uniquely determines each point. But a domain can be chosen about a point A such that a unique geodesic passes through the given point and any other point in the domain. * Consequently the above theorem is perfectly general.

From the above result we prove the theorem

Geodesics on a surface of negative curvature cannot meet in two points and inclose a simply connected area

Suppose that two geodesics through a point A pass through a point B , the two geodesics inclosing a simply connected area of the surface (fig 19). Take any geodesic cutting these

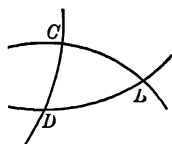


FIG 19

two segments AB in points C and D . Since the four angles $\angle ACD, \angle ADC, \angle BCD, \angle BDC$ are together equal to four right angles, the sum of the angles of the two triangles ADC and BDC exceeds four right angles by the sum of the angles at A and B . Therefore, in

consequence of the above theorem of Gauss, the total curvature of the surface cannot be negative at all points of the area $ADBC$. On the contrary, it can be shown that for a surface of positive curvature geodesics through a point meet again in general. In

fact, the exceptional points, if there are any, lie in a finite portion of the surface, which may consist of one or more simply connected parts * For example, the geodesics on a sphere are great circles, and all of these through a point pass through the diametrically opposite point Again, the helices are geodesics on a cylinder (§ 12), and it is evident that any number of them can be made to pass through two points at a finite distance from one another by varying the angle under which they cut the elements of the cylinder Hence the domain of a system of polar geodesic coordinates is restricted on a surface of positive curvature

89 Lines of shortest length Geodesic curvature We are now in a position to prove the theorem

If two points on a surface are such that only one geodesic passes through them, the segment of the geodesic measures the shortest distance on the surface between the two points

Take one of the points for the pole of a polar geodesic system and the geodesic for the curve $v=0$ The coordinates of the second point are $(u_1, 0)$ The parametric equation of any other curve through the two points is of the form $v=\phi(u)$, and the length of its arc is

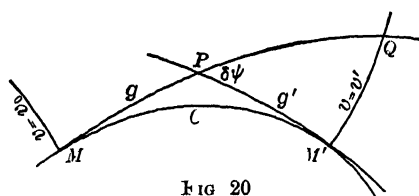


FIG 20

$$\int_0^{u_1} \sqrt{1 + G\phi'^2} du$$

Since $G > 0$, the value of this integral is necessarily greater than u_1 , and the theorem is proved

By means of equation (57) we derive another definition of geodesic curvature Consider two points M and M' upon a curve C , and the unique geodesics g, g' tangent to C at these points (fig 20) Let P denote the point of intersection of g and g' , and $\delta\psi$ the angle under which they cut Liouville † has called $\delta\psi$ the *angle of geodesic contingence*, because of its analogy to the ordinary angle of contingence Now we shall prove the theorem

The limit of the ratio $\delta\psi/\delta s$, as M' approaches M , is the geodesic curvature of C at M

* For a proof of this the reader is referred to a memoir by H v Mangoldt in *Crelle* Vol XCI (1881) pp 24-53

† *Journal de Mathématiques* Vol XVI (1851), p 132

In the proof of this theorem we take for parametric curves the curve C , its geodesic parallels and their geodesic orthogonals, the parameter u being the distance measured along the latter from C . If the geodesic g meets the curve $v = v_0$ orthogonally, the angle which it meets $v = v'$ may be denoted by $\pi/2 + \delta\theta$. As M' approaches M , $\delta\theta$ approaches $d\theta$ given by (57), and the sum of the angles of the triangle $M'PQ$ approaches 180° . Hence $\delta\psi$ approaches $d\psi$, so that we have

$$\lim \frac{\delta\psi}{\delta s} = - \frac{d\theta}{ds} = \frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial u},$$

which is the expression for the geodesic curvature of the curve C .
Geodesic ellipses and hyperbolas. An important system of geodesic lines for a surface is formed by two families of geodesic parallels. Such a system may be obtained by constructing the geodesic parallels of two curves C_1 and C_2 , which are not themselves geodesic parallels of one another, or by taking the two families of geodesic circles with centers at any two points F_1 and F_2 . Let u and v be the geodesic distances from C_1 and C_2 , or from F_1 and F_2 , respectively. They must be solutions of (53). Consequently, in terms of them, the linear element has the form

$$\frac{E}{EG - F^2} = \frac{G}{EG - F^2} = 1$$

usually, ω denotes the angle between these parametric lines, so that, from (III, 15, 16),

$$E = G = \frac{1}{\sin^2 \omega}, \quad F = \frac{\cos \omega}{\sin^2 \omega},$$

the linear element has the following form, due to Weingarten

$$ds^2 = \frac{du^2 + 2 \cos \omega \, du dv + dv^2}{\sin^2 \omega}$$

namely, when the linear element is reducible to this form, u and v are solutions of (53), and consequently the parametric curves are geodesic parallels.

In terms of the parameters u_1 and v_1 , defined by $u = u_1 + v_1$ and $v = u_1 - v_1$, the linear element (59) has the form

$$ds^2 = \frac{du_1^2}{\sin^2 \frac{\omega}{2}} + \frac{dv_1^2}{\cos^2 \frac{\omega}{2}}$$

The geometrical significance of the curves of parameter u_1 and v_1 is seen when the above equations are written

$$(61) \quad u_1 = \frac{1}{2}(u + v), \quad v_1 = \frac{1}{2}(u - v)$$

The curves $u_1 = \text{const}$ and $v_1 = \text{const}$ are respectively the loci of points the sum and difference of whose geodesic distances from C_1 and C_2 , or from F_1 and F_2 , are constant. In the latter case these curves are analogous to ellipses and hyperbolas in the plane, the points F_1 and F_2 corresponding to the foci. For this reason they are called *geodesic ellipses and hyperbolas*, which names are given likewise to the curves $u_1 = \text{const}$, $v_1 = \text{const}$, when the distances are measured from two curves, C_1 and C_2 . From (60) follows at once the theorem of Weingarten *

A system of geodesic ellipses and hyperbolas is orthogonal

By means of (61) equation (60) can be transformed into (59), thus proving that when the lineal element of a surface is in the form (60), the parametric curves are geodesic ellipses and hyperbolas.

If θ denotes the angle which the tangent to the curve $v_1 = \text{const}$ through a point makes with the curve $v = \text{const}$, it follows from (III, 23) that

$$\cos \theta = \cos \frac{\omega}{2}, \quad \sin \theta = \sin \frac{\omega}{2}$$

Hence we have the theorem

Given any two systems of geodesic parallels upon a surface, the corresponding geodesic ellipses and hyperbolas bisect the angles included by the former

91 Surfaces of Liouville Darb† inquired whether there were any surfaces with an isothermal system of geodesic ellipses and hyperbolas. A necessary and sufficient condition that such a surface exist is that the coefficients of (60) satisfy a condition of the form (§ 41)

$$V_1 \sin^2 \frac{\omega}{2} = U_1 \cos^2 \frac{\omega}{2},$$

where U_1 and V_1 denote functions of u_1 and v_1 respectively. In this case the lineal element may be written

$$(62) \quad ds^2 = (U_1 + V_1) \left(\frac{du_1^2}{U_1} + \frac{dv_1^2}{V_1} \right)$$

* Ueber die Oberflächen für welche einer der beiden Hauptkrümmungshalbmesser eine Function des anderen ist. *Crelle* Vol. L XII (1863) pp. 160-173.

† *Annali Sci.* 2 Vol. III (1869) pp. 269-293.

change of parameters defined by

$$u_2 = \int \frac{du_1}{\sqrt{U_1}}, \quad v_2 = \int \frac{dv_1}{\sqrt{V_1}},$$

each element is transformed into

$$ds^2 = (U_2 + V_2)(du_2^2 + dv_2^2),$$

U_2 and V_2 are functions of u_2 and v_2 respectively, such that

$$U_1(u_1) = U_2(u_2), \quad V_1(v_1) = V_2(v_2)$$

ely, if the linear element is in the form (63), it may be into (62) by the transformation of coordinates

$$u_1 = \int \sqrt{U_2} du_2, \quad v_1 = \int \sqrt{V_2} dv_2$$

ces whose linear element is reducible to the form (63) were died by Liouville, and on that account are called *surfaces of* e^* . To this class belong the surfaces of revolution and the (§§ 96, 97). We may state the above results in the form

the linear element of a surface is in the Liouville form, the principal curves are geodesic ellipses and hyperbolas, these systems only isothermal orthogonal families of geodesic conics †

Integration of the equation of geodesic lines. Having thus deduced the various properties of geodesic lines, and having seen the advantage of knowing their equations in finite form, we return to the consideration of their differential equation and derive certain results concerning its integration.

First, in the first place, that we know a particular first integral of the general equation, that is, a family of geodesics defined by the equation of the form

$$M du + N dv = 0$$

[IV, 58] it follows that M and N must satisfy the equation

$$\left(\frac{FN - GM}{\sqrt{EN^2 - 2FMN + GM^2}} \right) + \frac{\partial}{\partial v} \left(\frac{FM - EN}{\sqrt{EN^2 - 2FMN + GM^2}} \right) = 0$$

Journal de Mathématiques Vol XI (1846), p. 345

the reader is referred to Darboux Vol II p. 208, for a discussion of the conditions which a surface is of the Liouville type

In consequence of this equation we know that there exists a function ϕ defined by

$$(65) \quad \frac{\partial \phi}{\partial u} = \frac{EN - FM}{\sqrt{EN^2 - 2FMN + GM^2}}, \quad \frac{\partial \phi}{\partial v} = \frac{FN - GM}{\sqrt{EN^2 - 2FMN + GM^2}}$$

Moreover, we find that

$$(66) \quad \Delta_1 \phi = 1$$

From (III, 31) and (65) it follows that the curves $\phi = \text{const}$ are the orthogonal trajectories of the given geodesics, and from (66) it is seen that ϕ measures distance along the geodesics from the curve $\phi = 0$. Hence we have the theorem of Darboux *

When a one-parameter family of geodesics is defined by a differential equation of the first order, the finite equation of their orthogonal trajectories can be obtained by a quadrature, which gives the geodesic parameter at the same time

Therefore, when the general first integral of the equation of geodesics is known, all the geodesic parallels can be found by quadratures

We consider now the converse problem of finding the geodesics when the geodesic parallels are known. Suppose that we have a solution of equation (66) involving an arbitrary constant a , which is not additive. If this equation be differentiated with respect to a , we get

$$(67) \quad \Delta_1 \left(\phi, \frac{\partial \phi}{\partial a} \right) = 0,$$

where the differential parameter is formed with respect to the linear element. But this is a necessary and sufficient condition (§ 37) that the curves $\phi = \text{const}$ and the curves

$$(68) \quad \frac{\partial \phi}{\partial a} = \text{const} = a'$$

form an orthogonal system. Hence the curves defined by (68) are geodesics. In general, this equation involves two arbitrary constants, a and a' , which, as will now be shown, enter in such a way that this equation gives the general integral of the differential equation of geodesic lines

* *Leçons* Vol II p 430, cf also Bianchi, Vol I p 202

se that a appears in equation (68), and write the latter thus

$$\psi(u, v, a) = a',$$

case equation (67) becomes

$$\Delta_1(\phi, \psi) = 0$$

tion of each of the curves (69) is given by $\frac{\partial\psi}{\partial u} / \frac{\partial\psi}{\partial v}$. If this is independent of a , so also by (70) is the ratio $\frac{\partial\phi}{\partial u} / \frac{\partial\phi}{\partial v}$. The latter in the form

$$\frac{\partial\phi}{\partial u} = f(u, v) \frac{\partial\phi}{\partial v}$$

Equation and (66) be solved for $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$, we obtain values independent of a , so that a would have been additive. Hence f is independent of a , and so also does $\frac{\partial\psi}{\partial u} / \frac{\partial\psi}{\partial v}$, and therefore a direction at (u_0, v_0) determines the value of a , call it a_0 . If then a'_0 is that

$$\psi(u_0, v_0, a_0) = a'_0,$$

the geodesic $\psi(u, v, a_0) = a'_0$ passes through the point (u_0, v_0) and has the given direction at the point. Hence all the geodesics are determined by equation (68), and we have the theorem

a solution of the equation $\Delta_1\phi = 1$, involving an arbitrary constant a , in such a way that $\frac{\partial\phi}{\partial a}$ involves a , the equation

$$\frac{\partial\phi}{\partial a} = a',$$

*where a' is the finite equation of the geodesics, and the constant a is measured by ϕ **

From this result we establish the following theorem due to Darboux

For the first integral of the differential equation of geodesic lines to be expressible in finite equation can be found by one quadrature

The integral is of the form

$$\frac{dv}{du} = \psi(u, v, a),$$

where a is an arbitrary constant. As this equation is of the form (64), the function ϕ , defined by

$$\phi = \int \frac{(E + F\psi) du + (F + G\psi) dv}{\sqrt{E + 2F\psi + G\psi^2}},$$

is a solution of equation (66). As ϕ involves a in the manner specified in the preceding theorem, the finite equation of the geodesics is $\frac{\partial \phi}{\partial a} = a'$

93 Geodesics on surfaces of Liouville The surfaces of Liouville (§ 91) afford an excellent application of the theorem of Jacobi. We take the linear element in the form *

$$(71) \quad ds^2 = (U - V)(U_1^2 du^2 + V_1^2 dv^2),$$

which evidently is no more general than (63). In this case equation (66) becomes

$$\Delta_1 \phi = \frac{1}{U - V} \left[\frac{1}{U_1^2} \left(\frac{\partial \phi}{\partial u} \right)^2 + \frac{1}{V_1^2} \left(\frac{\partial \phi}{\partial v} \right)^2 \right] = 1$$

When this equation is written in the form

$$U - \frac{1}{U_1^2} \left(\frac{\partial \phi}{\partial u} \right)^2 = V + \frac{1}{V_1^2} \left(\frac{\partial \phi}{\partial v} \right)^2,$$

one sees that it belongs to the class of partial differential equations admitting an integral which is the sum of functions of u and v alone †. In order to obtain this integral, we put each side equal to a constant a and integrate. This gives

$$(72) \quad \phi = \int U_1 \sqrt{U - a} du \pm \int V_1 \sqrt{a - V} dv$$

Hence the equation of geodesics is

$$(73) \quad \frac{\partial \phi}{\partial a} = -\frac{1}{2} \int \frac{U_1}{\sqrt{U - a}} du \pm \frac{1}{2} \int \frac{V_1}{\sqrt{a - V}} dv = a'$$

If θ denotes the angle which a geodesic through a point makes with the line $v = \text{const}$ through the point, it follows from (III, 24) and (71) that

$$\tan \theta = \frac{V_1}{U_1} \frac{dv}{du}$$

* Cf Darboux Vol III p 9

† Forsyth *Differential Equations* (1888) p 310

value of dv/du from equation (73) be substituted in this, we obtain the following first integral of the Gauss equation (56)

$$U \sin^2 \theta + V \cos^2 \theta = a$$

which is due to Liouville *

EXAMPLES

1. On a surface of constant curvature the area of a geodesic triangle is proportional to the difference between the sum of the angles of the triangle and π angles.

2. Show that for a developable surface the first integral of equation (56) can be obtained by quadratures.

3. On any curve C upon a surface and the developable surface which is the envelope of the tangent planes to the surface along C , show that the geodesic curvature of C is equal to the curvature of the plane curve into which C is transformed when the developable is developed upon a plane.

4. On the plane is referred to a system of confocal ellipses and hyperbolas which are at the distance $2c$ apart, the linear element can be written

$$ds^2 = (u^2 - v^2) \left(\frac{du^2}{u^2 - c^2} + \frac{dv^2}{c^2 - v^2} \right)$$

Find the necessary and sufficient condition that ϕ be a solution of $\Delta_1 \phi = 1$ is that ϕ be a perfect square.

5. $\theta = \theta_1 \alpha + \theta_2$, where θ_1 and θ_2 are functions of u and v , is a solution of the equation $\Delta_1 \theta = 0$. The curves $\theta_1 = \text{const}$ are lines of length zero, and the curves $\theta_1 \alpha + \theta_2 = \text{const}$ are orthogonal trajectories.

6. On the linear element of a spiral surface is in the form $ds^2 = e^{2u} (du^2 + U^2 dv^2)$, where $U_1 \phi = 1$ admits the solution $e^u U_1$, where U_1 is a function of u , which satisfies an equation of the first order whose integration gives thus all the geodesics of the surface.

7. On a surface with the linear element

$$ds^2 = V [du^2 + (u + V_1)^2 dv^2],$$

where V_1 and V are functions of v alone, the equation $\Delta_1 \phi = 1$ admits the solution $\psi(u) + \psi_1(v)$, the determination of the functions ψ_1 and ψ requiring the solution of a differential equation of the first order and quadratures.

8. ϕ denotes a solution of $\Delta_1 \phi = 1$ involving a non-additive constant a , the metric of the surface can be written

$$ds^2 = d\phi^2 + \frac{\left(\phi, \frac{\partial \phi}{\partial a} \right)}{\left(\frac{\partial \phi}{\partial a}, \frac{\partial^2 \phi}{\partial a^2} \right)} \left(d \frac{\partial \phi}{\partial a} \right)^2,$$

ϕ, ψ indicates the mixed differential parameter (III, 48)

94 Lines of shortest length Envelope of geodesics We can go a step farther than the first theorem of § 89 and show that whether one or more geodesics pass through two points M_1 and M_2 on a surface, the shortest distance on the surface between these points, if it exists, is measured along one of these geodesics

Thus, let $v = f(u)$ and $v = f_1(u)$ define two curves C and C_1 passing through the points M_1, M_2 , the parametric values of u at the points being u_1 and u_2 . The arc of C between these points has the length

$$(75) \quad s = \int_{u_1}^{u_2} \sqrt{E + 2Fv' + Gv'^2} du,$$

where v' denotes the derivative of v with respect to u . For convenience we write the above thus

$$(76) \quad s = \int_{u_1}^{u_2} \phi(u, v, v') du$$

Furthermore, we put

$$f_1(u) = f(u) + \epsilon \omega(u),$$

where $\omega(u)$ is a function of u vanishing when u is equal to u_1 and u_2 , and ϵ is a constant whose absolute value may be taken so small that the curve C_1 will lie in any prescribed neighborhood of C . Hence the length of the arc M_1M_2 of C_1 is

$$s_1 = \int_{u_1}^{u_2} \phi(u, v + \epsilon \omega, v' + \epsilon \omega') du$$

Thus s_1 is a function of ϵ , reducing for $\epsilon = 0$ to s . Hence, in order that the curve C be the shortest of all the near-by curves which pass through M_1 and M_2 , it is necessary that the derivative of s_1 with respect to ϵ be zero for $\epsilon = 0$. This gives

$$\int_{u_1}^{u_2} \left(\frac{\partial \phi}{\partial v} \omega + \frac{\partial \phi}{\partial v'} \omega' \right) du = 0$$

On the assumption that ω admits a continuous first derivative in the interval (u_1, u_2) , and ϕ continuous first and second derivatives, the left hand member of this equation may be integrated by parts with the result

$$\int_{u_1}^{u_2} \omega \left(\frac{\partial \phi}{\partial v} - \frac{d}{du} \frac{\partial \phi}{\partial v'} \right) du = 0,$$

take $\omega(u) = \eta(u) \left(\phi_v - \frac{d}{du} \phi_{v'} \right) \quad \eta > 0$

nishes when u equals u_1 and u_2 . As the function ϕ is arbitrary for the above conditions upon it, this equation is not to the following equation of Euler *

$$\frac{\partial \phi}{\partial v} - \frac{d}{du} \frac{\partial \phi}{\partial v'} = 0$$

1 this result is applied to the particular form of ϕ in equation (43), we have

$$\frac{d}{du} \left(\frac{F + Gv'}{\sqrt{E + 2Fv' + Gv'^2}} \right) - \frac{\frac{\partial E}{\partial v} + 2 \frac{\partial F}{\partial v} v' + \frac{\partial G}{\partial v} v'^2}{2\sqrt{E + 2Fv' + Gv'^2}} = 0,$$

is readily reducible to equation (43)

is the shortest distance between two points, if existent, is found along a geodesic through the points. This geodesic is unique if the surface has negative total curvature at all points. On other surfaces more than one geodesic may pass through the points if they are sufficiently far apart. We now investigate the nature of this

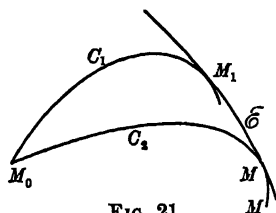


FIG 21

$= f(u, \alpha)$ define the family of geodesics through a point $M_0(u_0, v_0)$, and let α be the equation of their envelope.

We consider two of the geodesics C_1 and C_2 (fig 21), and (u_1, v_1) and $M_2(u, v_2)$ denote their points of contact with the envelope. Suppose that the arc M_0M_2 is greater than M_0M_1 . The distance from M_0 to M_2 , measured along C_1 and $\mathcal{C}$ is equal to

$$D = \int_{u_0}^{u_1} \phi(u, f, f') du + \int_{u_1}^u \phi(u, g, g') du$$

where u is considered fixed and M_1 variable, the position of the latter being determined by α . The variation of D with M_1 is given by

$$\int_{u_0}^{u_1} \left(\frac{\partial \phi}{\partial f} \frac{\partial f}{\partial \alpha} + \frac{\partial \phi}{\partial f'} \frac{\partial f'}{\partial \alpha} \right) du + \left[\phi(u, f, f') \frac{du_1}{d\alpha} - \phi(u, g, g') \frac{du_1}{d\alpha} \right]_{u=u_1}$$

modus invenendi lineas curvas maximè minimè proprietate gaudentes chap 11 (L'année, 1744), cf Bolza, *Lectures on the Calculus of Variations*, p 22 (Chicago,

But for $u = u_1$, $f = g$ and $f' = g'$, consequently the last term is zero. Integrating the first member by parts, and noting that $\frac{\partial f}{\partial \alpha}$ is zero for $u = u_0$ and $u = u_1$ (§ 26), we have

$$\frac{\partial D}{\partial \alpha} = \int_{u_0}^{u_1} \frac{\partial f}{\partial \alpha} \left(\frac{\partial \phi}{\partial f} - \frac{d}{du} \frac{\partial \phi}{\partial f'} \right) du$$

Since C_1 is a geodesic, the expression in parenthesis is zero, and hence D does not vary with M_1 . This shows that the envelope of the geodesics through a point bears to them the relation which the evolute of a curve does to a family of normals to the curve. Moreover, the curve $\mathcal{E}$ is not a geodesic, for at each point of it there is tangent a geodesic. Hence there is an arc connecting M_1 and M_2 which is shorter than the arc of $\mathcal{E}$. In this way, by taking different points M_1 on $\mathcal{E}$, we obtain any number of arcs connecting M_0 and M_2 which are shorter than the arc of C_2 , each consisting of an arc of a geodesic such as C_1 and the geodesic distance $M_1 M_2$. It is then necessarily true that the shortest distance from M_0 to a point M of C_2 beyond M_2 is not measured along C_2 . However, when M lies within the arc $M_0 M_2$, a domain can be chosen about C_2 so small that the arc $M_0 M$ of C_2 is shorter than the arc $M_0 M$ of any other curve within the domain and passing through these points*.

Another historical problem associated with this problem is the following †

Given an arc C_0 joining two points A, B on a surface, to find the curve of shortest length joining A and B , and inclosing with C_0 a given area

The area is given by $\iint H du dv$. It is evident that two functions M and N can be found in an infinity of ways such that

$$H = \frac{\partial N}{\partial u} - \frac{\partial M}{\partial v}$$

By the application of Green's theorem we have

$$\iint H du dv = \iint \left(\frac{\partial N}{\partial u} - \frac{\partial M}{\partial v} \right) du dv = \int M du + N dv$$

where the last integral is curvilinear and is taken around the contour of the area. Since C_0 is fixed, our problem reduces to the determination of a curve C' along which the integral $\int_A^B M du + N dv$ is constant, and whose arc AB , that is, the

* For a more complete discussion of this problem the reader is referred to Darboux Vol III pp 86-112. Bolza chap v.

† In fact it was in the solution of this problem that Minding (*Crelle* Vol V (1830) p 297) discovered the function to which Bonnet (*Journal de l'Ecole Polytechnique*, Vol XIX (1848), p 44) gave the name *geodesic curvature*.

$\int_A^B \sqrt{E + 2Fv' + Gv'^2} du$, is a minimum. From the calculus of variations that, so far as the differential equations of the solution is concerned, this is the same problem as finding the curve C along which the integral

$$\int_A^B \sqrt{E + 2Fv' + Gv'^2} du + c(M + Nv') du$$

is a minimum, c being a constant. Euler's equation for this integral is

$$\frac{d}{du} \left(\frac{F + Gv'}{\sqrt{E + 2Fv' + Gv'^2}} \right) - \frac{\frac{\partial E}{\partial v} + 2v \frac{\partial F}{\partial v} + v'^2 \frac{\partial G}{\partial v}}{\sqrt{E + 2Fv' + Gv'^2}} = cH$$

By comparing this result with the formula of Bonnet (IV, 56), we see that C has constant geodesic curvature $1/c$, and c evidently depends upon the magnitude of the area enclosed by the curves. Hence we have the theorem of Minding *

That a curve C joining two points shall be the shortest which, together with the geodesics through these points, incloses a portion of the surface with a given area, if the geodesic curvature of C be constant.

GENERAL EXAMPLES

Let the parametric curves on the unit sphere satisfy the condition

$$\frac{\partial}{\partial u} \left\{ \frac{12}{1} \right\}' = \frac{\partial}{\partial v} \left\{ \frac{12}{2} \right\}' = 2 \left\{ \frac{12}{1} \right\}' \left\{ \frac{12}{2} \right\}',$$

represent the asymptotic lines on a surface whose total curvature is

$$K = - \frac{1}{[\phi(u) + \psi(v)]^2}$$

Let the equations of the sphere have the form (III, 35), the parametric equations of the asymptotic and the equation (22) is $(1 + uv)^2 \frac{c^2 \theta}{\partial u \partial v} = -2\theta$, of which the integral is

$$\theta = 2 \frac{v\phi(u) + u\psi(v)}{1 + uv} - \phi(u) - \psi(v),$$

where ϕ and ψ denote arbitrary functions.

Let the sections of a surface by all the planes through a fixed line L in space be the curves of contact of the tangent cones to the surface whose vertices are on a conjugate system.

Let a surface of translation $x = u$, $y = v$, $z = f(u) + \phi(v)$. Determine the functions f and ϕ so that $(\rho_1 + \rho_2)Z = \text{const}$, where Z denotes the cosine of the angle which the normal makes with the z axis, and determine the lines of curvature on the surface.

Determine the relations between the exponents m , and n , in the equations

$$x = U^m V^{n_1}, \quad y = U^{m_2} V^n, \quad z = U^{m_3} V^{n_3},$$

show that the surface so defined the parametric curves shall form a conjugate system and show that the asymptotic lines can be found by quadratures.

6 The envelope of the family of planes

$$(U_1 + V_1)x + (U_2 + V_2)y + (U_3 + V_3)z + (U_4 + V_4) = 0,$$

where the U 's are functions of u alone and the V 's of v , is a surface upon which the parametric curves are plane, and form a conjugate system

7 The condition that the parametric curves form a conjugate system on the envelope of the plane

$$x \cos u + y \sin u + z \cot v = f(u, v),$$

is that f be the sum of a function of u alone and of v alone, in this case these curves are plane lines of curvature

8 Find the geodesics on the surface of Ex 7, p 219, and determine the expressions for the radii of curvature and torsion of a geodesic

9 A representation of two surfaces upon one another is said to be *conformal-conjugate* when it is at the same time conformal, and every conjugate system on one surface corresponds to a conjugate system on the other. Show that the lines of curvature correspond and that the characteristic lines also correspond

10 Given a surface of revolution $x = u \cos v$, $y = u \sin v$, $z = f(u)$, and the function ϕ defined by

$$(1) \quad \frac{1}{\phi'} + \sqrt{\frac{1}{\phi'^2} + 1} = A \left\{ \frac{1}{f'} + \sqrt{\frac{1}{f'^2} + 1} \right\}^{\frac{1}{c}},$$

where A and c are constants, a conformal-conjugate representation of the surface upon a second surface $x_1 = u_1 \cos v_1$, $y_1 = u_1 \sin v_1$, $z_1 = \phi(u_1)$ is defined by

$$v = cv_1, \quad c \log u_1 = \int \frac{\sqrt{1+f'^2}}{\sqrt{1+F'^2}} \frac{du}{u},$$

where F' denotes the function of u found by solving (1) for ϕ

11 If two families of geodesics cut under constant angle, the surface is developable

12 If a surface with the linear element

$$ds^2 = (au^2 - bv^2 - c)(du^2 + dv^2),$$

where a, b, c are constants, is represented on the xy -plane by $u = x$, $v = y$, the geodesics correspond to the Lissajous figures defined by

$$\sqrt{b} \sin^{-1} \frac{x}{A} - \sqrt{a} \sin^{-1} \frac{y}{B} = C,$$

where A, B, C are constants

13 When there is upon a surface more than one family of geodesics which, together with their orthogonal trajectories, form an isothermal system, the curvature of the surface is constant

14 If the principal normals of a curve meet a fixed straight line, the curve is a geodesic on a surface of revolution whose axis is this line. Examine the case where the principal normals meet the line under constant angle

representation of two surfaces upon one another is said to be a *geodesic* when to a geodesic on one surface there corresponds a geodesic on

Show that the representation is geodesic when points with the same c values correspond on surfaces with the linear elements

$$r - V)(U_1^2 du^2 + V_1^2 dv^2), \quad ds_1^2 = \left(\frac{1}{V+h} - \frac{1}{U+h} \right) \left(\frac{U_1^2 du^2}{U+h} + \frac{V_1^2 dv^2}{V+h} \right),$$

U 's are functions of u alone, the V 's of v alone, and h is a constant. surface with the linear element

$$ds^2 = (u^4 - v^4) \left[\phi \left(\frac{1}{u} \right) du^2 + \phi(v) dv^2 \right],$$

any function whatever, admits of a geodesic representation upon itself

necessary and sufficient condition that an orthogonal system upon a surface be regarded as geodesic ellipses and hyperbolas in two ways, is that when are parametric the linear element be of the Liouville form, in this case es may be so regarded in an infinity of ways

all the curves of equal length joining two points, the one which, together with curve through the points, incloses the area of greatest extent, has constant curvature

Let Γ be any curve upon a surface, and at two near-by points P, P' draw normals g, g' perpendicular to Γ , let C be the curve through P conjugate to the point where it meets g' , and Q the intersection of the tangents to g and P' , the limiting position of Q , as P' approaches P , is the center of curvature of Γ at P

Show that if a surface S admits of geodesic representation upon a plane in which four families of geodesics are represented by four families of parallel lines, each geodesic on the surface is represented by a straight line (cf Ex 3,

CHAPTER VII

QUADRICS RULED SURFACES MINIMAL SURFACES

95 Confocal quadrics Elliptic coordinates Two quadrics are *confocal* when the foci, real or imaginary, of their principal sections coincide. Hence a family of confocal quadrics is defined by the equation

$$(1) \quad \frac{x^2}{a^2 - u} + \frac{y^2}{b^2 - u} + \frac{z^2}{c^2 - u} = 1,$$

where u is the parameter of the family and a, b, c are constants, such that

$$(2) \quad a^2 > b^2 > c^2$$

For each value of u , positive or negative, less than a^2 , equation (1) defines a quadric which is

$$(3) \quad \begin{cases} \text{an ellipsoid when } c^2 > u > -\infty, \\ \text{an hyperboloid of one sheet when } b^2 > u > c^2, \\ \text{an hyperboloid of two sheets when } a^2 > u > b^2 \end{cases}$$

As u approaches c^2 the smallest axis of the ellipsoid approaches zero. Hence the surface $u = c^2$ is the portion of the xy plane, counted twice, bounded by the ellipse

$$(4) \quad \frac{x^2}{a^2 - c^2} + \frac{y^2}{b^2 - c^2} = 1, \quad z = 0$$

Again, the surface $u = b^2$ is the portion of the xz plane, counted twice, bounded by the hyperbola

$$(5) \quad \frac{x^2}{a^2 - b^2} - \frac{z^2}{b^2 - c^2} = 1, \quad y = 0,$$

which contains the center of the curve. Equations (4) and (5) define the *focal ellipse* and *focal hyperbola* of the system

Through each point (x, y, z) in space there pass three quadrics similarly, they are determined by the values of u , which are the equation

$$\phi(u) = (a^2 - u)(b^2 - u)(c^2 - u) - x^2(b^2 - u)(c^2 - u) - y^2(a^2 - u)(c^2 - u) - z^2(a^2 - u)(b^2 - u) = 0$$

$$\phi(a^2) < 0, \quad \phi(b^2) > 0, \quad \phi(c^2) < 0, \quad \phi(-\infty) > 0,$$

the roots of equation (6), denoted by u_1, u_2, u_3 , are contained in the following intervals

$$a^2 > u_1 > b^2, \quad b^2 > u_2 > c^2, \quad c^2 > u_3 > -\infty$$

From it is seen that the surfaces corresponding to u_1, u_2, u_3 are respectively hyperboloids of two and one sheets and an ellipsoid. Fig. 22 represents three confocal quadrics, the curves on the

surfaces are lines of curvature. On the hyperboloid of one sheet they are principal lines

From the definition of u_1 , it follows that $\phi(u)$ is $(u_1 - u)(u_2 - u)(u_3 - u)$. In (6) $\phi(u)$ is replaced by this expression and u is successively the a^2, b^2, c^2 , we obtain *

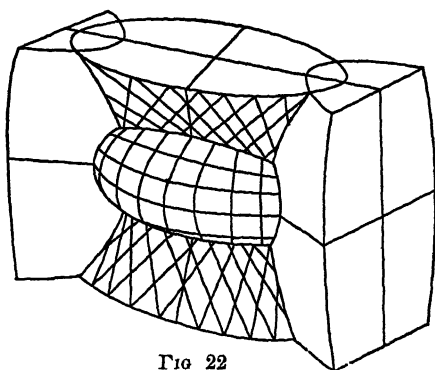


FIG. 22

$$\begin{cases} x = \frac{(a^2 - u_1)(a^2 - u)(a^2 - u_3)}{(a^2 - b^2)(a^2 - c^2)}, \\ y = \frac{(b^2 - u_1)(b^2 - u)(b^2 - u_3)}{(b^2 - a^2)(b^2 - c^2)}, \\ z = \frac{(c^2 - u_1)(c^2 - u)(c^2 - u_3)}{(c^2 - a^2)(c^2 - b^2)} \end{cases}$$

These formulas express the Cartesian coordinates of a point in space in terms of the parameters of the three quadrics which pass through the point. These parameters are called the *elliptic coordinates* of the point. It is evident that to each set of these

* Knudhoff *Mechanik*, p. 203. Leipzig 1877.

ordinates there correspond eight points in space, one in each of the eight compartments bounded by the coordinate planes. If one of the parameters u_i in (8) be made constant, and the others u_j, u_k , where $i \neq j \neq k$, be allowed to vary, these equations define in parametric form the surface, also defined by equation (10), in which u has this constant value u_i . The parametric curves $u_j = \text{const}, u_k = \text{const}$ are the curves of intersection of the given quadric and the double system of quadrics corresponding to the parameters u_j and u_k . If we put

$$1) \quad a^2 - u_i = a, \quad b^2 - u_i = b, \quad c^2 - u_i = c, \quad u_j - u_i = u, \quad u_k - u_i = v,$$

the equation of the surface becomes

$$0) \quad \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1,$$

and the parametric equations (8) reduce to

$$(11) \quad \begin{cases} x = \sqrt{\frac{a(a-u)(a-v)}{(a-b)(a-c)}}, \\ y = \sqrt{\frac{b(b-u)(b-v)}{(b-a)(b-c)}}, \\ z = \sqrt{\frac{c(c-u)(c-v)}{(c-a)(c-b)}} \end{cases}$$

Moreover, the quadrics which cut (10) in the parametric curves have the equations

$$(12) \quad \begin{cases} \frac{x^2}{a-u} + \frac{y^2}{b-u} + \frac{z^2}{c-u} = 1, \\ \frac{x^2}{a-v} + \frac{y^2}{b-v} + \frac{z^2}{c-v} = 1 \end{cases}$$

In consequence of (3) and (9) we have that equations (10) or (11) define

$$(13) \quad \begin{cases} \text{an ellipsoid when } a > u > b > v > c > 0, \\ \text{an hyperboloid of one sheet when } a > u > b > 0 > c > v, \\ \text{an hyperboloid of two sheets when } a > 0 > b > u > c > v \end{cases}$$

undamental quantities for central quadrics By direct calculation we find from (11)

$$E = \frac{u(u-v)}{f(u)}, \quad F = 0, \quad G = \frac{v(v-u)}{f(v)},$$

or the sake of brevity we have put

$$f(\theta) = 4(a-\theta)(b-\theta)(c-\theta)$$

we also the following

$$\begin{cases} X = \sqrt{\frac{bc(a-u)(a-v)}{uv(a-b)(a-c)}}, \\ Y = \sqrt{\frac{ca(b-u)(b-v)}{uv(b-a)(b-c)}}, \\ Z = \sqrt{\frac{ab(c-u)(c-v)}{uv(c-a)(c-b)}} \end{cases}$$

$$D = -\sqrt{\frac{abc}{uv}} \frac{u-v}{f(u)}, \quad D' = 0, \quad D'' = \sqrt{\frac{abc}{uv}} \frac{u-v}{f(v)}$$

and D' are zero, the parametric curves are lines of curvature and since the change of parameters (9) did not change the principal curves, we have the theorem

quadrics of a confocal system cut one another along lines of curvature, and the three surfaces through a point cut one another orthogonally at the point

result is illustrated by fig 22

(14) and (17) we have

$$\frac{1}{\rho_1} = -\sqrt{\frac{abc}{u^3v}}, \quad \frac{1}{\rho_2} = -\sqrt{\frac{abc}{uv^3}}, \quad \frac{1}{\rho_1\rho_2} = \frac{abc}{u^2v^2}$$

the ellipsoid and hyperboloid of two sheets have positive curvature at all points, whereas the curvature is negative at all points of the hyperboloid of one sheet

formula (16) be written

$$X = \sqrt{\frac{abc}{uv}} \frac{x}{a}, \quad Y = \sqrt{\frac{abc}{uv}} \frac{y}{b}, \quad Z = \sqrt{\frac{abc}{uv}} \frac{z}{c},$$

hence W from the center to the tangent plane is

$$W = \sum Xx = \sqrt{\frac{abc}{uv}}$$

Hence

The tangent planes to a central quadric along a curve, at points of which the total curvature of the surface is the same, are equally distant from the center

From (18) we see that the umbilical points correspond to the values of the parameters such that $u = v$. The conditions (18) show that this common value of u and v for an ellipsoid is b , and c for an hyperboloid of two sheets, whereas there are no real umbilical points for the hyperboloid of one sheet. When these values are substituted in (11), we have as the coordinates of these points on the ellipsoid

$$(20) \quad x = \pm \sqrt{\frac{a(a-b)}{(a-c)}}, \quad y = 0, \quad z = \pm \sqrt{\frac{c(b-c)}{(a-c)}},$$

and on the hyperboloid of two sheets

$$(21) \quad x = \pm \sqrt{\frac{a(a-c)}{(a-b)}}, \quad y = \pm \sqrt{\frac{b(b-c)}{(b-a)}}, \quad z = 0$$

It should be noticed that these points lie on the focal hyperbola and focal ellipse respectively

97 Fundamental quantities for the paraboloids The equation of a paraboloid

$$(22) \quad 2z = ax^2 + by^2$$

may be replaced by

$$(23) \quad x = \sqrt{u_1}, \quad y = \sqrt{v_1}, \quad z = \frac{1}{2}(au_1 + bv_1)$$

Hence the paraboloids are surfaces of translation (§ 81) whose generating curves are parabolas which lie in perpendicular planes. By direct calculation we find

$$E = \frac{1}{4} \left(\frac{1}{u_1} + a \right), \quad F = \frac{1}{4} ab, \quad G = \frac{1}{4} \left(\frac{1}{v_1} + b \right),$$

$$D = \frac{1}{4} \frac{a}{u_1 \sqrt{a^2 u_1 + b^2 v_1 + 1}}, \quad D' = 0, \quad D'' = \frac{1}{4} \frac{b}{v_1 \sqrt{a^2 u_1 + b^2 v_1 + 1}},$$

so that the equation of the lines of curvature is

$$v_1 = u_1 \frac{dv_1}{du_1} + \left[\frac{b-a}{a^2 b} \frac{dv_1}{du_1} - \frac{b}{a} v_1 \frac{dv_1}{du_1} + \frac{b}{a} u_1 \left(\frac{dv_1}{du_1} \right)^2 \right]$$

eral integral of this equation is

$$v_1 = c \left[u_1 + \frac{b-a}{a^2 b} - \frac{b}{a} v_1 \right] + \frac{b}{a} u_1 c^2,$$

is an arbitrary constant

1 u_1 and v_1 in (24) are given particular values, equation (24) has two values of c , c_1 and c_2 , in general distinct. If these values be substituted in (24) successively, we obtain in finite equations of the two lines of curvature through the point

If c_1 and c_2 be replaced by $-\left(\frac{1+au}{bu}\right)$ and $-\left(\frac{1+av}{bv}\right)$ respectively, we have, in consequence of (23), the two equations

$$\begin{cases} buy^2 + (1+au)x^2 = u(1+au) \frac{b-a}{ab}, \\ bvy^2 + (1+av)x^2 = v(1+av) \frac{b-a}{ab} \end{cases}$$

These equations are solved for x^2 and y^2 , we find that equation (23) can be replaced by

$$\begin{cases} x^2 = \frac{a-b}{b} uv, \\ y^2 = \frac{b-a}{ab^2} (1+au)(1+av), \\ z = \frac{1}{2} \frac{b-a}{ab} (1+au+av), \end{cases}$$

parametric curves are the lines of curvature we have

$$\begin{cases} E = \frac{a-b}{4b^2} (u-v) \frac{a(a-b)u-b}{u(1+au)}, & F = 0, \\ G = \frac{b-a}{4b} (u-v) \frac{a(a-b)v-b}{v(1+av)}, \end{cases}$$

$$Y, Z = \frac{a^2 \sqrt{a-b} \sqrt{uv}, \sqrt{b} \sqrt{b-a} \sqrt{(1+au)(1+av)}, -\sqrt{ab}}{\sqrt{[a(a-b)u-b][a(a-b)v-b]}},$$

$$D = \frac{1}{4} \sqrt{\frac{a^3}{b}} \frac{(a-b)(u-v)}{\sqrt{[a(a-b)u-b][a(a-b)v-b]}} \frac{1}{u(1+au)},$$

$$D' = 0,$$

$$D'' = -\frac{1}{4} \sqrt{\frac{a^3}{b}} \frac{(a-b)(u-v)}{\sqrt{[a(a-b)u-b][a(a-b)v-b]}} \frac{1}{v(1+av)}$$

From (27), (28), and (29) we obtain

$$(30) \quad W = \sum Xx = \frac{\sqrt{ab}z}{[a(a-b)u-b]^{\frac{1}{2}}[a(a-b)v-b]^{\frac{1}{2}}},$$

and

$$(31) \quad \begin{cases} \frac{1}{\rho_1} = \sqrt{a^3b^3}[a(a-b)u-b]^{-\frac{3}{2}}[a(a-b)v-b]^{-\frac{1}{2}}, \\ \frac{1}{\rho_2} = \sqrt{a^3b^3}[a(a-b)u-b]^{-\frac{1}{2}}[a(a-b)v-b]^{-\frac{3}{2}} \end{cases}$$

From these results we find that the ratio W/z is constant along the curves for which the total curvature is constant

We suppose that b is positive and greater than a . From the first of (26) it follows that u and v at a real point differ in sign, or one is equal to zero. We consider the points at which both u and v are equal to zero. There are two such points, and their coordinates are

$$(32) \quad x=0, \quad y = \pm \frac{1}{b} \sqrt{\frac{b-a}{a}}, \quad z = \frac{1}{2} \frac{b-a}{ab}$$

Evidently these points are real only on the elliptic paraboloid. From (31) it follows that ρ_1 and ρ_2 are then equal, and consequently these are the umbilical points. Since at points other than these u and v must differ in sign, we may assume that u is always positive and v negative. Moreover, from (26) it is seen that u and v are unrestricted except in the case of the elliptic paraboloid, when v must be greater than $-1/a$.

98 Lines of curvature and asymptotic lines on quadrics From (14), (27), and § 91 we have the theorem

The lines of curvature of a quadric surface form an isothermal system of the Liouville type

Bonnet* has shown that this property is characteristic of the quadrics. There are, however, many surfaces whose lines of curvature form an isothermal system. They are called *isothermic surfaces*. The complete determination of all such surfaces has never been accomplished (cf Ex 3, § 65)

* Mémoire sur la théorie des surfaces applicables sur une surface donnée, *Journal de l'Ecole Polytechnique*, Vol XXV (1867), pp 121-132

(17), (29), and § 82 follows the theorem

lines of curvature of a quadric surface form an isothermal system, and consequently the asymptotic lines can be found

we shall find the expressions for the coordinates in terms of u and v in another way

Equation (10) is equivalent to the pair of equations

$$\left(\frac{x}{\sqrt{a}} + i\frac{z}{\sqrt{c}}\right) = u\left(1 - \frac{y}{\sqrt{b}}\right), \quad \left(\frac{x}{\sqrt{a}} - i\frac{z}{\sqrt{c}}\right) = \frac{1}{u}\left(1 + \frac{y}{\sqrt{b}}\right),$$

and

$$\left(\frac{x}{\sqrt{a}} + i\frac{z}{\sqrt{c}}\right) = \frac{1}{v}\left(1 + \frac{y}{\sqrt{b}}\right), \quad \left(\frac{x}{\sqrt{a}} - i\frac{z}{\sqrt{c}}\right) = v\left(1 - \frac{y}{\sqrt{b}}\right),$$

and v are undetermined. For each value of u equations (10) define a line all of whose points lie on the surface. And to each point on the surface there corresponds a value of u determined through the point. Hence the surface is ruled, and it is doubly ruled, as seen from (18). Again, for each value of v equation (34) defines a line whose points lie on the surface. These lines are different from those of the other system. Hence the central quadrics are doubly ruled. These lines are precisely the asymptotic lines. Consequently, if equations (33), solved for x , y , z , thus

$$\frac{x}{\sqrt{a}} = \frac{u+v}{uv+1}, \quad \frac{y}{\sqrt{b}} = \frac{uv-1}{uv+1}, \quad \frac{z}{\sqrt{c}} = i\frac{v-u}{uv+1},$$

the surface defined in terms of parameters referring to the asymptotic lines

the manner equation (22) may be replaced by

$$\begin{aligned} \sqrt{ax} + i\sqrt{by} &= 2uz, & \sqrt{ax} - i\sqrt{by} &= \frac{1}{u}, \\ \sqrt{ax} + i\sqrt{by} &= \frac{1}{v}, & \sqrt{ax} - i\sqrt{by} &= 2vz \end{aligned}$$

hence, we have

$$\sqrt{ax} = \frac{u+v}{2uv}, \quad \sqrt{by} = \frac{i(v-u)}{2uv}, \quad z = \frac{1}{2uv}$$

As in the preceding case, we see that the surface is doubly ruled,* and the parameters in (36) refer to the asymptotic system of straight lines. Hence

The asymptotic lines on any quadric are straight lines

EXAMPLES

4 The focal conics of a family of confocal quadrics meet the latter in the points.

the characteristic lines on the quadrics of positive curvature

5 The normal section of an ellipsoid at a point in the direction of the curve along which the total curvature is constant is an ellipse with one of its vertices at the point.

6 Find the equation of the form $\frac{\partial^2 \theta}{\partial u \partial v} = M\theta$ (cf § 79) when the corresponding surface is a hyperboloid of one sheet, when a hyperbolic paraboloid

7 Find the evolute of the hyperboloid of one sheet and derive the following properties

(a) the surface is algebraic of the twelfth order,

(b) the section by a principal plane of the hyperboloid consists of a conic and the evolute of a conic,

(c) these sections are edges on the surface,

(d) the curve of intersection of the two sheets of the surface is cut by each of the principal planes in four ordinary points, four double points, and four cusps, and consequently is of the twenty fourth order

8 Determine for the evolute of a hyperbolic paraboloid the properties analogous to those for the surface of Ex. 5

9 Deduce the equations of the surfaces parallel to a central quadric, determine their order and the character of the sections of the surface by the principal planes of the quadric, find the normal curvature of the curves corresponding to the asymptotic lines on the quadric

99 Geodesics on quadrics Since the quadrics are isothermic surfaces of the Liouville type, the finite equation of the geodesics can be found by quadratures (§ 93). From (VI, 74), (14) and (27),

* Moreover the quadrics are the only doubly ruled surfaces. For consider such a surface, and denote by a, b, c three of the generators in one system. A plane α through a meets b and c in unique points B and C and the line BC meets a in a point 1. The line ABC is a generator of the second system, and the only one of this system in the plane α . The other lines of this system meet α in the line a . On this account the plane α cuts the surface in two lines, a and ABC , that is, in a degenerate conic. Hence the surface is of the second degree.

us that the first integral of the differential equation of s on any one of the quadrics is

$$u \sin^2 \theta + v \cos^2 \theta = \alpha,$$

α is a constant of integration and θ measures the angle the geodesic, determined by a value of α , makes with the lines of curvature $v = \text{const.}$ We recall that in equations (11) and (26) the parameter u is greater than v , except at the umbilical points, where they are equal. We shall discuss the general case first.

Under a particular point $M'(u', v')$ According as α is given between u' or v' , equation (37) defines the geodesic tangent at M' to the line of curvature $u = u'$ or $v = v'$ respectively. It is seen that the other values of α , determining other geodesics through M' , lie in the interval between u' and v' . Moreover, each value of α in this domain there correspond two geodesics through M' whose tangents are symmetrically placed with respect to the directions of the lines of curvature. From all this it follows also that the whole system of geodesics is determined by (37), when α is given the limiting values of u and v and the intermediate values.

We write equation (37) in the form

$$(u - \alpha) \sin^2 \theta + (v - \alpha) \cos^2 \theta = 0,$$

Consider the geodesics on a central quadric defined by this equation when α has a particular value α' . Suppose, first, that $\alpha' > v$. Then at each point of these geodesics $v < \alpha'$ and consequently from (38) $u > \alpha'$. We have seen that these geodesics are tangent to the line of curvature $u = \alpha'$. From (11) it follows that they lie within the zone of the surface bounded by the two branches of the curve $u = \alpha'$. When, now, $\alpha' < v$, the domain of the values of v , $u - \alpha'$ is positive, and consequently from (38) $v < \alpha'$. Hence the geodesics tangent to the line of curvature $v = \alpha'$ lie outside the zone bounded by the two branches of the curve of curvature $v = \alpha'$. Similar results are true for the paraboloids with the difference, as seen from (26), that the geodesics tangent to $u = \alpha'$ lie outside the region bounded by this curve, and the curves tangent to $v = \alpha'$ lie inside the region bounded by this curve.

x'

100 Geodesics through the umbilical points There remains for consideration the case where α takes the unique value which u and v have at the umbilical points. Let it be denoted by α_0 , so that the curves defined by

$$(39) \quad (u - \alpha_0) \sin^2 \theta + (v - \alpha_0) \cos^2 \theta = 0$$

are the *umbilical geodesics*. We have, at once, the theorem

Through each point on a quadric with real umbilical points there pass two umbilical geodesics which are equally inclined to the lines of curvature through the point

Hence two diametrically opposite umbilical points of an ellipsoid are joined by an infinity of geodesics, and no two geodesics through the same umbilical point meet again except at the diametrically opposite point. These properties are possessed also by a family of great circles on a sphere through two opposite points. On the elliptic paraboloid and on each sheet of the hyperboloid of two sheets there are two families of umbilical geodesics, but no two of the same family meet except at the umbilical point common to all curves of the family.

For the ellipsoid (11) $\alpha_0 = b$ and equations (VI, 72, 73) become

$$\phi = \frac{1}{2} \int \sqrt{\frac{u}{(a-u)(u-c)}} du \pm \frac{1}{2} \int \sqrt{\frac{v}{(a-v)(v-c)}} dv,$$

$$\frac{\partial \phi}{\partial b} = -\frac{1}{4} \int \sqrt{\frac{u}{(a-u)(u-c)}} \frac{du}{u-b} \mp \frac{1}{4} \int \sqrt{\frac{v}{(a-v)(v-c)}} \frac{dv}{v-b}$$

Similar results hold for the hyperboloid of two sheets and the elliptic paraboloid. Hence the distances of a point P from two umbilical points (not diametrically opposite) are of the form

$$\phi_1 = f_1(u) + f_2(v), \quad \phi_2 = f_1(u) - f_2(v)$$

Hence we have

The lines of curvature on the quadrics with real umbilical points are geodesic ellipses and hyperbolas with the umbilical points for foci

101 Ellipsoid referred to a polar geodesic system A family of umbilical geodesics and their orthogonal trajectories constitute an excellent system for polar geodesic coordinates, because the domain is unrestricted (§ 87) except in the case of the ellipsoid,

and then only the diametrically opposite point must be excluded. We consider such a system on the ellipsoid, and let O denote the pole of the system and O' , O'' , O''' the other umbilical points (fig 23)

If we put

$$(10) \quad \begin{cases} \phi = \frac{1}{2} \int \sqrt{\frac{u}{(a-u)(u-c)}} du - \frac{1}{2} \int \sqrt{\frac{v}{(a-v)(v-c)}} dv, \\ \psi = \frac{1}{2} \int \sqrt{\frac{u}{(a-u)(u-c)}} \frac{du}{u-b} - \frac{1}{2} \int \sqrt{\frac{v}{(a-v)(v-c)}} \frac{dv}{v-b}, \end{cases}$$

it is readily found that

$$\Delta_1 \phi = 1, \quad \Delta_1 (\phi, \psi) = 0, \quad \Delta_1 \psi = -\frac{1}{(b-u)(b-v)}$$

By means of (11) we may reduce the linear element to the form

$$(11) \quad ds^2 = d\phi^2 + \frac{(a-b)(b-c)}{b} \psi^2 d\psi^2$$

In order that the coordinates be polar geodesic, ψ must be replaced by another parameter measuring the angles between the geodesics. For the ellipsoid equation (39) is

$$(12) \quad (u-b) \sin^2 \theta + (v-b) \cos^2 \theta = 0$$

As previously seen, θ is half of one of the angles between the two geodesics through a point M . As M approaches O along the geodesic joining these two points, the geodesic $O'MO'''$ ap-

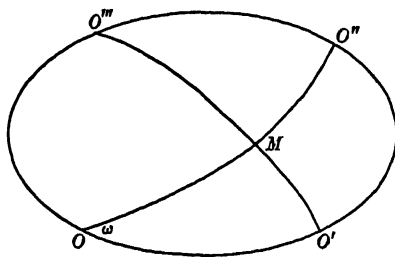


FIG. 23

proaches the section $y=0$. Consequently the angle 2θ approaches the angle MOO' , denoted by ω , or its supplementary angle. Hence we have from (42)

$$(43) \quad \lim_{u=b, v=b} \left(\frac{b-v}{u-b} \right) = \tan^2 \frac{\omega}{2}$$

We take ω in place of ψ and indicate the relation between them by $\psi = f(\omega)$. From (41) we have

$$\sqrt{G} = \sqrt{(u-b)(b-v)} f'(\omega)$$

This expression satisfies the first of conditions (VI, 54) The second is

$$(44) \quad \lim_{\substack{u=b \\ v=b}} \frac{1}{2} \frac{f'(\omega)}{\sqrt{(u-b)(b-v)}} \left[(b-v) \frac{\partial u}{\partial \phi} - (u-b) \frac{\partial v}{\partial \phi} \right] = 1$$

If we make use of the formulas (III, 11) and (40), we find

$$\frac{\partial u}{\partial \phi} = 2 \sqrt{\frac{(a-u)(u-c)}{u}} \frac{u-b}{u-v}, \quad \frac{\partial v}{\partial \phi} = -2 \sqrt{\frac{(a-v)(v-c)}{v}} \frac{b-v}{u-v},$$

so that equation (44) reduces to

$$\lim_{\substack{u=b \\ v=b}} f'(\omega) \frac{\sqrt{(u-b)(b-v)}}{u-v} \left[\sqrt{\frac{(a-u)(u-c)}{u}} + \sqrt{\frac{(a-v)(v-c)}{v}} \right] = 1$$

By means of (43) we pass from this to

$$(45) \quad f'(\omega) = \sqrt{\frac{b}{(a-b)(b-c)}} \frac{1}{\sin \omega}$$

Hence the linear element has the following form due to Roberts *

$$(46) \quad ds^2 = d\phi^2 + \frac{y^2}{\sin^2 \omega} d\omega^2$$

The second of equations (40) may now be put in the form

$$\begin{aligned} & \frac{1}{2} \int \sqrt{\frac{u}{(a-u)(u-c)}} \frac{du}{u-b} - \frac{1}{2} \int \sqrt{\frac{v}{(a-v)(v-c)}} \frac{dv}{v-b} \\ & = \sqrt{\frac{b}{(a-b)(b-c)}} \log \tan \frac{\omega}{2} + C, \end{aligned}$$

where C denotes the constant of integration. In order to evaluate this constant, we consider the geodesic through the point $(0, b, 0)$. At this point the parameters have the values $u = a$, $v = c$, and the angle ω has a definite value $\bar{\omega}$. Hence the above equation may be replaced by

$$\begin{aligned} & \frac{1}{2} \int_a^u \sqrt{\frac{u}{(a-u)(u-c)}} \frac{du}{u-b} - \frac{1}{2} \int_c^v \sqrt{\frac{v}{(a-v)(v-c)}} \frac{dv}{v-b} \\ & = \sqrt{\frac{b}{(a-b)(b-c)}} \log \left(\frac{\tan \frac{\omega}{2}}{\tan \frac{\bar{\omega}}{2}} \right) \end{aligned}$$

manner, for the umbilical geodesics through one of the foci (not diametrically opposite) we have

$$\int_a^u \sqrt{\frac{u}{(a-u)(u-c)}} \frac{du}{u-b} + \frac{1}{2} \int_c^v \sqrt{\frac{v}{(a-v)(v-c)}} \frac{dv}{v-b} \\ = \sqrt{\frac{b}{(a-b)(b-c)}} \log \left(\frac{\tan \frac{\omega}{2}}{\tan \frac{\bar{\omega}}{2}} \right)$$

It follows at once from these formulas that if M is any point on a line of curvature $u = \text{const}$ or $v = \text{const}$, we have respectively

$$\frac{OO'}{2} \tan \frac{MO'O}{2} = \text{const}, \quad \tan \frac{MOO'}{2} \cot \frac{MO'O}{2} = \text{const}$$

Properties of quadrics From (18) it follows that for the quadrics Euler's equation (IV, 34) takes the form

$$\frac{1}{R} = -(v \cos^2 \theta + u \sin^2 \theta) \sqrt{\frac{abc}{u^3 v^3}}$$

From (19) and (37) this reduces to

$$\frac{1}{R} = -\alpha \frac{W^3}{abc}$$

In the same manner, we have for the paraboloids

$$\frac{1}{R} = -\frac{W^3}{z^3} [b + \alpha \alpha (b - a)]$$

From (38) we have

On a geodesic or line of curvature on a central quadric the ratio RW^3 is constant, and on a paraboloid the ratio RW^3/z^3 is constant.

Consider any point P on a central quadric and a direction at P . Let α, β, γ be the direction cosines of the latter with respect to the principal diameters of the ellipsoid (10) parallel to this direction is given by

$$\frac{1}{\rho^2} = \frac{\alpha^2}{a} + \frac{\beta^2}{b} + \frac{\gamma^2}{c}$$

Then

$$\alpha = \frac{\cos \theta}{\sqrt{E}} \frac{\partial x}{\partial u} + \frac{\sin \theta}{\sqrt{G}} \frac{\partial x}{\partial v},$$

and similarly for β and γ . When the values of x, y, z, E, G from (11) and (14) are substituted, equation (49) reduces to

$$\frac{1}{\rho^2} = \frac{\cos^2 \theta}{u} + \frac{\sin^2 \theta}{v}$$

By means of (19) and (37) this may be reduced to

$$(50) \quad \alpha \rho^2 W^2 = abc$$

From this follows the theorem of Joachimsthal

Along a geodesic or a line of curvature on a central quadric the product of the semi-diameter of the quadric parallel to the tangent to the curve at a point P and the distance from the center to the tangent plane at P is constant

From (47) and (50) we obtain the equation

$$\rho^2 = -RW,$$

or all points on the quadric. Since W is the same for all directions at a point, the maximum and minimum values of ρ and R correspond. Hence we have the theorem

*In the central section of a quadric parallel to the tangent plane at a point P the principal axes are parallel to the directions of the lines of curvature at P **

EXAMPLES

1 On a hyperbolic paraboloid, of which the principal parabolas are equal, the locus of a point, the sum or difference of whose distances from the generators through the vertex of the paraboloid is constant, is a line of curvature.

2 Find the radii of curvature and torsion, at the extremity of the mean diameter of an ellipsoid, of an umbilical geodesic through the point.

3 Find the surfaces normal to the tangents to a family of umbilical geodesics on an ellipsoid, and determine the complementary surface (cf. § 76).

4 The geodesic distance of two diametrically opposite umbilical points on an ellipsoid is equal to one half the length of the principal section through the umbilical points.

5 Find the form of the linear element of the hyperboloid of two sheets or the elliptic paraboloid, when the parametric system is polar geodesic with an umbilical point for pole.

6 If M_1 and M_2 are two points of intersection of a geodesic through the umbilical point O with a line of curvature $v = \text{const}$, then

$$\tan \frac{M_1 O' O}{2} \cot \frac{M_2 O' O}{2} = \text{const}$$

* For a more complete discussion of the geodesics on quadrics, the reader is referred to a memoir by v. Braunmühl, in *Math. Annalen*, Vol. XX (1882), pp. 556-586.

ven a line of curvature on an ellipsoid and the geodesics tangent to it, a of intersection of pairs of these geodesics, meeting orthogonally, lie on

ven the geodesics tangent to two lines of curvature, the points of intersection of pairs of these geodesics, meeting orthogonally, lie on a sphere.

Equations of a ruled surface A surface which can be generated by the motion of a straight line is called a *ruled surface*. Developables are ruled surfaces for which the lines, called the *generators*, are tangent to a curve. As a general thing, ruled surfaces do not possess this property, and in this case they are called *surfaces*. Now we make a direct study of ruled surfaces, particularly those of the skew type, limiting our discussion to the case where the generators are real.*

A ruled surface is completely determined by a curve upon it and the direction of the generators at their points of meeting with the curve. We call the latter the *directrix*. If we draw through the cone formed by drawing through the directrix lines parallel to the generators the *ruled cone*. If the coordinates of a point D are x_0, y_0, z_0 , expressed in terms of the arc v measured along the directrix, and l, m, n are the direction-cosines of the generator through M_0 , the equations of the surface are

$$x = x_0 + lu, \quad y = y_0 + mu, \quad z = z_0 + nu,$$

u is the distance from M_0 to a point M on the generator through M_0 . If θ_0 denotes the angle which the generator through M_0 makes with the tangent at M_0 to D , then

$$\cos \theta_0 = x'_0 l + y'_0 m + z'_0 n,$$

the accent indicates differentiation with respect to v (fig. 24). (51) we find for the linear element the expression

$$ds^2 = du^2 + 2 \cos \theta_0 du dv + (a^2 u^2 + 2 bu + 1) dv^2,$$

we have put for the sake of brevity

$$\begin{cases} a^2 = l'^2 + m'^2 + n'^2, \\ b = l'x'_0 + m'y'_0 + n'z'_0 \end{cases}$$

* We shall use the term *ruled* to specify the surfaces of the skew type, and *developable* for the others.

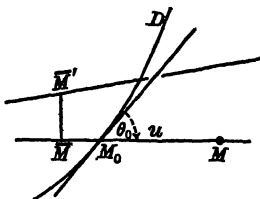


FIG. 24

Since the generators are geodesics, their orthogonal trajectories can be found by quadratures (§ 92). We arrive at this result directly by remarking that the equation of these trajectories is II, 26)

$$du + \cos \theta_0 dv = 0,$$

and that θ_0 is a function of v alone

104 Line of striction Developable surfaces We shall now consider the quantities which determine the relative positions of the generators of a ruled surface

Let g and g' be two generators determined by parametric values v and $v + \delta v$, and let λ, μ, ν denote the direction-cosines of their common perpendicular. If the direction-cosines of g and g' be denoted by $l, m, n, l + \delta l, m + \delta m, n + \delta n$ respectively, we have

$$5) \quad \begin{cases} l\lambda + m\mu + n\nu = 0, \\ (l + \delta l)\lambda + (m + \delta m)\mu + (n + \delta n)\nu = 0, \end{cases}$$

and consequently

$$\lambda \quad \mu \quad \nu = (m\delta n - n\delta m) \quad (n\delta l - l\delta n) \quad (l\delta m - m\delta l)$$

from (54) it follows that

$$(mn' - nm')^2 + (nl' - ln')^2 + (lm' - ml')^2 = \alpha^2,$$

and by Taylor's theorem,

$$7) \quad l + \delta l = l + l'\delta v + \frac{l''}{2} \delta v^2 +$$

hence equations (56) may be replaced by

$$8) \quad \begin{cases} \lambda = \frac{1}{a} (mn' - m'n) + \epsilon_1, \\ \mu = \frac{1}{a} (nl' - n'l) + \epsilon_2, \\ \nu = \frac{1}{a} (lm' - l'm) + \epsilon_3, \end{cases}$$

where $\epsilon_1, \epsilon_2, \epsilon_3$ denote expressions of the first and higher orders in δv . If $\bar{M}(\bar{x}, \bar{y}, \bar{z})$ and $\bar{M}'(\bar{x} + \delta\bar{x}, \bar{y} + \delta\bar{y}, \bar{z} + \delta\bar{z})$ are the points of meeting of this common perpendicular with g and g' respectively (§ 24), the length $\bar{M}\bar{M}'$, denoted by Δ , is given by

$$9) \quad \Delta = \frac{\delta\bar{x}}{\lambda} = \frac{\delta\bar{y}}{\mu} = \frac{\delta\bar{z}}{\nu},$$

$$10) \quad \Delta = \lambda\delta\bar{x} + \mu\delta\bar{y} + \nu\delta\bar{z}$$

1), after the manner of (57), we obtain

$$\delta\bar{x} = (x'_0 + \bar{u}l')\delta v + l\delta\bar{u} + \sigma,$$

involves the second and higher powers of δv . When similar values for $\delta\bar{y}$ and $\delta\bar{z}$ are substituted in (60), we have

$$\frac{\Delta}{\delta v} = p + \epsilon,$$

$$p = \frac{1}{a} \begin{vmatrix} x'_0 & y'_0 & z'_0 \\ l & m & n \\ l' & m' & n' \end{vmatrix},$$

involves first and higher powers of δv . In consequence of (54) we have

$$p^2 = \frac{a^2 \sin^2 \theta_0 - b^2}{a^2}$$

As δv approaches zero, the point $\bar{M}$ approaches a limiting position which is called the *central point* of the generator. Let α be the value of u for this point. In order to find its value we note that it follows from the equations (55) and (59) that

$$\frac{\delta\bar{x}}{\delta v} \frac{\delta l}{\delta v} + \frac{\delta\bar{y}}{\delta v} \frac{\delta m}{\delta v} + \frac{\delta\bar{z}}{\delta v} \frac{\delta n}{\delta v} = 0$$

The above expressions for these quantities be substituted in this equation, we have in the limit, as δv approaches zero,

$$a^2\bar{u} + b = 0$$

Consequently

$$\alpha = -\frac{b}{a},$$

The locus of the central points is called the *line of striction*. Its intrinsic equation is (64). Evidently $b=0$ is a necessary and sufficient condition that the line of striction be the directrix of the surface. In (61) and (63) it is seen that the distance between neighboring generators is of the second order when

$$a^2 \sin^2 \theta_0 - b^2 = 0$$

In the loss of generality we may take the line of striction to be the directrix, in which case we may have $\sin \theta_0 = 0$, that is, the

generators are tangent to the directrix. Another possibility is afforded by $a = 0$. From (54) it is seen that the only real surfaces satisfying this condition are cylinders. Hence (cf § 4)

A necessary and sufficient condition that a ruled surface, other than a cylinder, be developable is that the distance between near-by generators be of the second or higher orders, in this case the edge of regression is the line of striction.

105 Central plane **Parameter of distribution** The tangent plane to a ruled surface at a point M necessarily contains the generator through M . It has been found (§ 25) that for a developable surface this plane is tangent at all points of the generator.

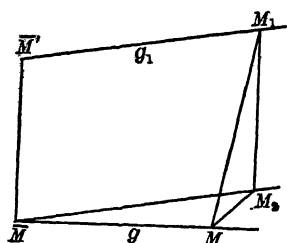


FIG. 25

We shall see that in the case of skew surfaces the tangent plane varies as M moves along the generator. We determine the character of this variation by finding the angle which the tangent plane at M makes with the tangent plane at the central point C of the generator through M . The tangent plane at C is called the *central plane*.

Let g and g_1 be two generators, and $\overline{MM'}$ their common perpendicular (fig. 25). Through the point M of g draw the plane normal to g , it meets g_1 in M_1 , and the line through $\overline{M}$ parallel to g_1 in M_2 . The limiting positions of the planes $M_1M\overline{M}$ and $\overline{M'}MM$, as g_1 approaches g , are the tangent planes at M and at C , the limiting position of $\overline{M}$. The angle between these planes, denoted by ϕ' , is equal to $\angle MM_1M_2$, and the angle between g and g_1 , denoted by σ , is equal to $\angle \overline{MM}M_2$. By construction $\angle MM_1M_2$ and $\angle \overline{MM}M_2$ are right angles. Hence

$$\tan \phi' = \frac{MM_1}{M_1M_2} = \frac{M\overline{M} \tan \sigma}{\overline{M}\overline{M}'}$$

In the limit $\overline{M}$ is the central point C , and so we have

$$\tan \phi = \lim \tan \phi' = \frac{(u - \alpha) d\sigma}{p dv} = \pm \frac{(u - \alpha) a}{p},$$

for we have $d\sigma^2 = \lim (\delta l^2 + \delta m^2 + \delta n^2) = a^2 dv^2$

tomary to write the above equation in the form

$$\tan \phi = \frac{u - \alpha}{\beta}$$

ction β thus defined is called the *parameter of distribution* a limit of the ratio of the shortest distance between two rs and their included angle As it is independent of the er u , we have the theorem

angent of the angle between the tangent plane to a ruled at a point M and the central plane is proportional to the of M from the central point

thus it follows that as M moves along a generator from $-\infty$ ϕ varies from $-\pi/2$ to $\pi/2$ Hence the tangent planes at utely distant points are perpendicular to the central plane $= 0$ is the condition that a surface be developable, the plane is the same at all points of the generator

all now derive equation (67) analytically From (51) we find direction-cosines of the normal to the surface are of the form

$$X = \frac{(mz'_0 - ny'_0) + (mn' - m'n)u}{(a^2u^2 + 2bu + \sin^2\theta_0)^{\frac{1}{2}}},$$

pressions for Y and Z are similar to the above The a-cosines X_0, Y_0, Z_0 of the normal at the central point are l from these by replacing u by α From this we have

os $\phi = \Sigma XX_0$

$$= \frac{\Sigma (mz'_0 - ny'_0)^2 + \Sigma (mz'_0 - ny'_0)(mn' - m'n)(u + \alpha) + a^2u\alpha}{(a^2u^2 + 2bu + \sin^2\theta_0)^{\frac{1}{2}}(a'^2a^2 + 2b\alpha + \sin^2\theta_0)^{\frac{1}{2}}},$$

eads to

$$\tan^2 \phi = \frac{a^4(u - \alpha)^2}{a'^2 \sin^2 \theta_0 - b}$$

us equation and (67) we have

$$\beta = \pm \frac{\sqrt{a^2 \sin^2 \theta_0 - b^2}}{a^2} = \pm \frac{1}{a^2} \begin{vmatrix} x'_0 & y'_0 & z'_0 \\ l & m & n \\ l' & m' & n' \end{vmatrix}$$

he surface is defined by its linear element, β is thus deter- nely to within an algebraic sign We shall find, however, s is not the case when the surface is defined by equations orm (51)

To this end we take a particular generator g for the z -axis for g we have

$$x_0 = y_0 = 0, \quad l = m = 0, \quad n = 1, \quad n' = 0$$

Let also the central plane be taken for the xz -plane and the central point for the origin. From (68) it follows that $y'_0 = 0$. Since the origin is the central point, $b = 0$ and consequently $l' = 0$. Hence the equation of the tangent plane at a point of g has the simple form

$$(71) \quad m'u\xi - x'_0\eta = 0,$$

ξ and η being current coordinates. If the coordinate axes have the usual orientation, and the angle ϕ is measured positively in the direction from the positive x -axis to the positive y -axis, from equation (71) we have

$$(72) \quad \tan \phi = \frac{m'u}{x'_0}$$

Comparing this with equation (67), we find for β the value x'_0/m' . In order to obtain the same value from (70) for these particular values, we must take the negative sign. Hence we have, in general,

$$(73) \quad \beta = -\frac{1}{a^2} \begin{vmatrix} x'_0 & y'_0 & z'_0 \\ l & m & n \\ l' & m' & n' \end{vmatrix}$$

It is seen from (72) that, as a point moves along a generator in the direction of u increasing, the motion of the tangent plane is that of a right-handed or left handed screw, according as β is negative or positive.

EXAMPLES

- 1 Show that for the ruled surface defined by

$$x = \frac{1}{2} \int (1 - u^2) \phi \, du + \frac{1}{2} (1 - v^2) v,$$

$$y = \frac{i}{2} \int (1 + u^2) \phi \, du + \frac{i}{2} (1 + v^2) v,$$

$$z = \int u \phi \, du + \psi v,$$

where ϕ and ψ are any functions of u , the directrix and the generators are minimal. Determine under what condition the curvature of the surface is constant.

- 2 Determine the condition that the directrix of a ruled surface be a geodesic.

re, by means of (62), that the lines of curvature of a surface $F(x, y, z) = 0$ d by

$$\begin{vmatrix} dx, & dy, & dz \\ \frac{\partial F}{\partial x}, & \frac{\partial F}{\partial y}, & \frac{\partial F}{\partial z} \\ d\frac{\partial F}{\partial x}, & d\frac{\partial F}{\partial y}, & d\frac{\partial F}{\partial z} \end{vmatrix} = 0$$

right helicoid is the only ruled surface whose generators are the principal f their orthogonal trajectories Find the parameter of distribution

re for the hyperboloid of revolution of one sheet that

- (a) the minimum circle is the line of striction and a geodesic,
- (b) the parameter of distribution is constant.

h every point P on a ruled surface there is associated another point P' ie generator, such that the tangent planes at these points are perpendicular t the product $OP \cdot OP'$, where O denotes the central point, has the same all points P on the same generator

normals to a ruled surface along a generator form a hyperbolic paraboloid

cross-ratio of four tangent planes to a ruled surface at points of a gen- equal to the cross-ratio of the points

wo ruled surfaces are symmetric with respect to a plane, the values of eter of distribution for homologous generators differ only in sign

Particular form of the linear element A number of prop- ruled surfaces are readily obtained when the linear element a particular form, which we will now deduce

n orthogonal trajectory of the generators be taken for the

In this case

$$\theta_0 = \frac{\pi}{2}, \quad \beta^2 = \frac{a^2 - b^2}{a^4}, \quad \alpha = -\frac{b}{a^2}$$

ake the change of parameters,

$$u = u, \quad v_1 = \int_0^v a \, dv,$$

ar element (53) is reducible to

$$ds^2 = du^2 + [(u - \alpha)' + \beta'] dv_1^2$$

le θ which a curve $v_1 = f(u)$ makes with the generators is y

$$\tan \theta = \sqrt{(u - \alpha) + \beta'} f'$$

e expression for the total curvature is

$$K = -\frac{\beta^2}{[(u - \alpha)^2 + \beta^2]^2}$$

Hence a real ruled surface has no elliptic points. All the points are hyperbolic except along the generators for which $\beta = 0$, and at the infinitely distant points on each generator. Consequently the linear element of a developable surface may be put in the form

$$(79) \quad ds^2 = du^2 + (u - \alpha)^2 dv^2$$

Also, in the region of the infinitely distant points of a ruled surface the latter has the character of a developable surface. As another consequence of (78) we have that, for the points of a generator the curvature is greatest in absolute value at the central point, and that at points equally distant from the latter it has the same value.

When the linear element is in the form (76), the Gauss equation of geodesics (VI, 56) has the form

$$\sqrt{(u - \alpha)^2 + \beta^2} d\theta + (u - \alpha) dv_1 = 0$$

An immediate consequence is the theorem of Bonnet

If a curve upon a ruled surface has two of the following properties, it has the third also, namely that it cut the generators under constant angle, that it be a geodesic and that it be the line of striction

A surface of this kind is formed by the family of straight lines which cut a twisted curve under constant angle and are perpendicular to its principal normals. A particular case is the surface formed of the binormals of a curve. It is readily shown from (73) that the parameter of distribution of this surface is equal to the radius of torsion of the curve.

107 Asymptotic lines. Orthogonal parametric systems. The generators are necessarily asymptotic lines on a ruled surface. We consider now the other family of these lines. From (51) and (68) we find

$$(80) \quad D = 0, \quad D' = \frac{1}{H} \begin{vmatrix} l' & m' & n' \\ l & m & n \\ x'_0 & y'_0 & z'_0 \end{vmatrix}, \quad D'' = \frac{1}{H} \begin{vmatrix} x''_0 + l''u & l & x'_0 + l'u \\ y''_0 + m''u & m & y'_0 + m'u \\ z''_0 + n''u & n & z'_0 + n'u \end{vmatrix}$$

Hence the differential equation of the other family of asymptotic lines is of the form

$$\frac{du}{dv} + Lu^2 + Mu + N = 0,$$

, M , N are functions of v . As this is an equation of the type, we have, from § 14, the theorem of Serret

our points in which each generator of a ruled surface is cut by curved asymptotic lines are in constant cross ratio

§ 14 it follows also that when one of these asymptotic lines is known the others can be found by quadratures

the surface is referred to an orthogonal system and the linear element is in (76), written

$$ds^2 = du^2 + \alpha^2 [(u - \alpha)^2 + \beta^2] dv^2,$$

equations (80) can be given a simpler form (78) and (81) we have

$$H = \alpha [(u - \alpha)^2 + \beta^2]^{\frac{1}{2}}, \quad D' = \frac{\alpha^2 \beta}{H}$$

equations $\Sigma x_0 l = 0$, $\Sigma x_0^2 = 1$, $\Sigma l^2 = 1$,

we obtain, by differentiation,

$$\begin{aligned} \Sigma x_0' x_0' &= 0, & \Sigma l l' &= 0, & \Sigma x_0' l &= -b, \\ \Sigma l' l' &= \alpha \alpha', & \Sigma l l'' &= -\alpha^2, & \Sigma l x_0 &= b' - t, \end{aligned}$$

defined by $\Sigma l' x_0' = t$

expression for D' in (80) be multiplied by the determinant of the right-hand member of (78), and the result be divided by its equal, $-\alpha^2 \beta$, we have, in consequence of the above identities,

$$D'' = -\frac{1}{H \alpha^2 \beta} [u^2 (t \alpha^2 - \alpha \alpha' b) + u (2 t b - \alpha \alpha' - b b') + t - b']$$

equations (74) be solved for a and b as functions of α and β , and the resulting values be substituted in this equation, we have

$$D = -\frac{\alpha^3}{H} \{ r [(u - \alpha)^2 + \beta^2] + \beta' (u - \alpha) + \beta \alpha' \},$$

primes indicate differentiation with respect to v , given by (70), and r by

$$\alpha^2 \beta r = t - \frac{\alpha}{a} b$$

the above equations it follows that the mean curvature (cf. § 52) is expressed in the form

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = -\frac{r [(u - \alpha)^2 + \beta^2] + \beta' (u - \alpha) + \beta \alpha'}{[(u - \alpha)^2 + \beta^2]^{\frac{3}{2}}}$$

EXAMPLES

When the linear element of a ruled surface is in the form (76), the directions of the limiting position of the common perpendicular to two generators are

$$\frac{x_0 + \alpha l'}{\alpha \beta}, \quad \frac{y_0 + \alpha m'}{\alpha \beta}, \quad \frac{z_0 + \alpha n}{\alpha \beta}$$

2 Prove that the developable surfaces are the only ruled surfaces with real generators whose total curvature is constant

3 Show that the perpendicular upon the z -axis from any point of the cubic $x = u$, $y = u^2$, $z = u^3$ lies in the osculating plane at the point, and find the asymptotic lines on the ruled surface generated by this perpendicular

4. Determine the function ϕ in the equations

$$x = u, \quad y = u^2, \quad z = \phi(u),$$

so that the osculating plane at any point M of this curve shall pass through the projection P of M on the y axis. Find the asymptotic lines on the surface generated by the line MP

5 Show that the equations

$$x = u \sin \theta \cos \psi, \quad y = u \sin \theta \sin \psi, \quad z = v + u \cos \theta,$$

where θ and ψ are functions of v , define the most general ruled surface with a rectilinear directrix, and prove that the equation of asymptotic lines can be integrated by two quadratures. Discuss the case where θ is constant

6 Concerning the curved asymptotic lines on a ruled surface the following are to be proved

(a) if one of them is an orthogonal trajectory of the generators, the determination of the rest reduces to quadratures,

(b) if two of them are orthogonal trajectories, they are curves of Bertrand,

(c) if all of them are orthogonal trajectories, the surface is a right helicoid

7 Determine the condition that the line of striction be an asymptotic line, and show that in this case the other curved asymptotic lines can be found by quadratures

8 Find a ruled surface of the fourth degree which is generated by a line passing through the two lines $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$. Show that these lines and the line $x = 0$, $x + y + z = 1$ are double lines. Find the line of striction

9 The right helicoid is the only ruled surface each of whose lines of curvature cuts the generators under constant angle, however, on any other ruled surface there are in general four lines of curvature which have this property

108 Minimal surfaces In 1760 Lagrange extended to double integrals the Euler theorems about simple integrals in the calculus of variations, and as an example he proposed the following problem*

Given a closed curve C and a connected surface S bounded by the curve, to determine S so that the inclosed area shall be a minimum

If the surface be defined by the equation

$$z = f(x, y),$$

the problem requires the determination of $f(x, y)$ so that the integral (cf Ex 1, p 77)

$$\iint \sqrt{1 + p^2 + q^2} dx dy$$

* *Œuvres de Lagrange*, Vol I pp 354-357 Paris, 1867

d over the portion of the surface bounded by C shall be a minimum. As shown by Lagrange, the condition for this is

$$\frac{\partial}{\partial x} \left(\frac{p}{\sqrt{1+p^2+q^2}} \right) + \frac{\partial}{\partial y} \left(\frac{q}{\sqrt{1+p^2+q^2}} \right) = 0,$$

in other form,

$$(1+q^2)r - 2pq s + (1+p^2)t = 0$$

We left the solution of the problem in this form, and in 1826, sixteen years later, proved that this equation is equivalent to the vanishing of the mean curvature (§ 52), thus indicating that the surfaces furnishing the solution of Lagrange's problem are characterized by the geometrical property which now is usually taken as the definition of *minimal surfaces*, however, the discovery indicates the connection with the definition of Lagrange.† In what follows we purpose giving a discussion of minimal surfaces from the standpoint of their definition as the surfaces whose mean curvature is zero at all the points. At each point of such a surface the principal radii differ only in sign, and so every point is a hyperbolic point and its Dupin indicatrix is an equilateral hyperbola. Consequently minimal surfaces are characterized by the property that their asymptotic lines form an orthogonal system. Moreover, the tangents to the two asymptotic lines at a point bisect the angles between the lines of curvature at the point, and vice versa.

We recall the formulas giving the relations between the fundamental quantities of a surface and its spherical representation

$$\mathcal{E} = K_m D - K E, \quad \mathcal{F} = K_m D' - K F, \quad \mathcal{G} = K_m D'' - K G$$

whence we have at once the theorem

A necessary and sufficient condition that the spherical representation of a surface be conformal is that the surface be minimal

Sur la courbure des surfaces, *Memoires des Savants étrangers*, Vol. X, p. 477.

A historical sketch of the development of the theory of minimal surfaces and a discussion of them the reader is referred to the *Lectures* of Darboux (Vol. I, pp. 101-110). The questions in the calculus of variations involved in the study of minimal surfaces are treated by Riemann, *Gesammelte Werke*, p. 287 (Leipzig 1876) and by Weierstrass, *Gesammelte Abhandlungen*, Vol. I, pp. 223, 270 (Berlin, 1890).

Hence isothermal orthogonal systems on the surface are represented by similar systems on the sphere, and conversely. All isothermal orthogonal systems on the sphere are known (§§ 35, 36). Suppose that one of these systems is parametric and that the line element is

$$d\sigma^2 = \lambda (du^2 + dv^2)$$

From the general condition for minimal surfaces (IV, 77), namely

$$(86) \quad \mathcal{E}D'' + \mathcal{G}D - 2\mathcal{F}D' = 0,$$

it follows that in this case $D + D'' = 0$

In consequence of this the Codazzi equations (V, 27) are reducible to

$$(87) \quad \frac{\partial D}{\partial v} - \frac{\partial D'}{\partial u} = 0, \quad \frac{\partial D}{\partial u} + \frac{\partial D'}{\partial v} = 0$$

By eliminating D or D' we find that both D and D' are integrals of the equation

$$\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} = 0$$

Hence the most general form of D' is

$$(88) \quad D' = \phi(u+v) + \psi(u-v),$$

where ϕ and ψ are arbitrary functions. Then from (87) we have

$$(89) \quad D = -D'' = -c(\phi - \psi) + c,$$

where c is the constant of integration. To each pair of functions ϕ, ψ there corresponds a minimal surface whose Cartesian coordinates are given by the quadratures (V, 26), namely

$$(90) \quad \frac{\partial x}{\partial u} = -\frac{1}{\lambda} \left(D \frac{\partial X}{\partial u} + D' \frac{\partial X}{\partial v} \right), \quad \frac{\partial x}{\partial v} = -\frac{1}{\lambda} \left(D' \frac{\partial X}{\partial u} + D'' \frac{\partial X}{\partial v} \right),$$

and similar expressions in y and z . Evidently the surface is planar only when ϕ and ψ are conjugate functions.

In obtaining the preceding results we have tacitly assumed that neither D nor D' is zero. We notice that either may be zero, or then the other is a constant, which is zero only for the planar case. These results may be stated thus

Every isothermal system on the sphere is the representation of lines of curvature of a unique minimal surface and of the asymptotic lines of another minimal surface.

In like manner we may take, for the solution of equations (87),
 (92) $D = D'' = 0, \quad D' = 1$

Again we find

$$\frac{1}{\rho_1 \rho_2} = -\frac{1}{\rho_1^2} = -\lambda^2, \quad E = G = \rho,$$

so that we have a result similar to the above

The parameters of the asymptotic lines of a minimal surface may be so chosen that the linear elements of the surface and of its spherical representation have the respective forms

$$ds^2 = \rho (du^2 + dv^2), \quad d\sigma^2 = \frac{1}{\rho} (du^2 + dv^2),$$

where ρ is the absolute value of each principal radius

From the symmetric form of equations (87) it follows that if (88) and (89) represent one set of solutions, another set is given by

$$D_1 = -D_1'' = \phi + \psi, \quad D_1' = i(\phi - \psi) - c$$

These values are such that

$$DD_1'' + D''D_1 - 2D'D_1' = 0,$$

which is the condition that asymptotic lines on either surface correspond to a conjugate system on the other (§ 56). When this condition is satisfied by two minimal surfaces, and the tangent planes at corresponding points are parallel, the two surfaces are said to be the *adjoints* of one another. Hence a pair of functions ϕ, ψ determines a pair of *adjoint minimal surfaces*. When, in particular, the asymptotic lines on one surface are parametric, the functions have the values (92), and on the other the values (91). It follows, then, from (90), that between the Cartesian coordinates of a minimal surface and its adjoint the following relations hold

$$(93) \quad \frac{\partial x_1}{\partial u} = -\frac{\partial x}{\partial v}, \quad \frac{\partial x_1}{\partial v} = \frac{\partial x}{\partial u},$$

and similar expressions in the y 's and z 's, when the parametric curves are asymptotic on the locus of (x, y, z)

110 Minimal curves on a minimal surface The lines of length zero upon a minimal surface are of fundamental importance. When they are taken for parametric curves, the equations of the surface take a simple form, which we shall now obtain

the lines of length zero, or minimal lines, are parametric,

$$E = G = 0$$

5) it follows that the parametric lines on the sphere also are lines, that is, the imaginary rectilinear generators. And 3) we find that D' is zero. Conversely, when the latter is the parametric lines are minimal curves, it follows from 1) that K_m is equal to zero. Hence

*necessary and sufficient condition that a surface be minimal is lines of length zero form a conjugate system.**

In consequence of (94) and (VI, 26) the point equation of a surface, referred to its minimal lines, is

$$\frac{\partial^2 \theta}{\partial u \partial v} = 0$$

The finite equations of the surface are of the form

$$x = U_1 + V_1, \quad y = U + V_2, \quad z = U_3 + V_3,$$

U_1, U_2, U_3 are functions of u alone, and V_1, V_2, V_3 are functions of v alone, satisfying the conditions

$$U_1'^2 + U_2'^2 + U_3'^2 = 0, \quad V_1'^2 + V_2'^2 + V_3'^2 = 0$$

95) it is seen that minimal surfaces are surfaces of translation (§ 81), and from (96) that the generators are minimal (§ 22). In consequence of the second theorem of § 81 we state this result thus

A minimal surface is the locus of the mid points of the joins of corresponding points on two minimal curves

22 we found that the Cartesian coordinates of any minimal surface are expressible in the form

$$\int (1 - u^2) F(u) du, \quad i \int (1 + u^2) F(u) du, \quad 2 \int u F(u) du$$

It follows also from the fact that an equilateral hyperbola is the only conic for which the directions with angular coefficients $\pm i$ are conjugate

Hence by the above theorem the following equations, due to Enneper *, define a minimal surface referred to its minimal lines

$$(98) \quad \begin{cases} x = \frac{1}{2} \int (1-u^2) F(u) du + \frac{1}{2} \int (1-v^2) \Phi(v) dv, \\ y = \frac{i}{2} \int (1+u^2) F(u) du - \frac{i}{2} \int (1+v^2) \Phi(v) dv, \\ z = \int u F(u) du + \int v \Phi(v) dv, \end{cases}$$

where F and Φ are any analytic functions whatever. Moreover, any minimal surface can be defined by equations of this form. For, the only apparent lack of generality is due to the fact that the algebraic signs of the expressions (98) are not determined by equations (96), and consequently the signs preceding the terms in the right-hand members of equations (98) could be positive or negative. But it can be shown that by a suitable change of the parameters and of the functions F and Φ all of these cases reduce to (98). Thus, for example, we consider the surface defined by the equations which result when the second terms of the right-hand members of (98) are replaced by

$$\frac{1}{2} \int (1-v_1^2) \Phi_1(v_1) dv_1, \quad \frac{i}{2} \int (1+v_1^2) \Phi_1(v_1) dv_1, \quad \int v_1 \Phi_1(v_1) dv_1$$

In order that the surface thus defined can be brought into coincidence, by a translation, with the surface (98), we must have

$$(1-v_1^2) \Phi_1 dv_1 = (1-v^2) \Phi dv, \quad (1+v_1^2) \Phi_1 dv_1 = -(1+v^2) \Phi dv, \\ v_1 \Phi_1 dv_1 = v \Phi dv$$

Dividing these equations, member by member, we have

$$\frac{1-v_1^2}{1-v^2} = -\frac{1+v_1^2}{1+v^2} = \frac{v_1}{v},$$

from which it follows that

$$v_1 = -\frac{1}{v}$$

Substituting this value in the last of the above equations, we find

$$\Phi_1(v_1) = -v^4 \Phi(v),$$

* *Zeitschrift für Mathematik und Physik*, Vol. IX (1864) p. 107

value satisfies the other equations. Similar results follow on another choice of signs is made. The reason for the particular choice made in (98) will be seen when we discuss the properties of the surfaces.

Finally we have proved the theorem

a minimal surface is defined by equations (98), the necessary sufficient condition that the two generating curves be congruent

$$F(u) = -\frac{1}{u^2} \Phi\left(-\frac{1}{u}\right)$$

(98) we obtain

$$E = 0, \quad F = \frac{1}{2}(1+uv)^2 F(u) \Phi(v), \quad G = 0,$$

the linear element is

$$ds^2 = (1+uv)^2 F(u) \Phi(v) du dv$$

and for the expressions of the direction-cosines of the normal

$$X = \frac{u+v}{1+uv}, \quad Y = \frac{v-u}{1+uv}, \quad Z = \frac{uv-1}{1+uv},$$

the linear element of the sphere is

$$d\sigma^2 = \frac{4 du dv}{(1+uv)^2}$$

we have

$$D = -F(u), \quad D' = 0, \quad D'' = -\Phi(v),$$

the equations of the lines of curvature and of the asymptotes are respectively

$$F(u) du^2 - \Phi(v) dv^2 = 0,$$

$$F(u) du^2 + \Phi(v) dv^2 = 0$$

the equations are of such a form that we have the theorem

if a minimal surface is referred to its minimal lines, the finite parts of the lines of curvature and asymptotic lines are given by circles, which are the same in both cases

For that a surface be real its spherical representation must

Consequently u and v must be conjugate imaginaries, as

is seen from (101) and § 13, and the functions F and Φ must be conjugate imaginary. Hence if $R\theta$ denotes the real part of a function θ , all real minimal surfaces are defined by

$$\begin{aligned}x &= R \int (1 - u^2) F(u) du, & y &= R \int i(1 + u^2) F(u) du, \\z &= R \int 2u F(u) du,\end{aligned}$$

where $F(u)$ is any function whatever of a complex variable u . In like manner the equations of the lines of curvature may be written in the form

$$(105) \quad R \int \sqrt{F(u)} du = \text{const}, \quad R \int i \sqrt{F(u)} du = \text{const}$$

111 Double minimal surfaces It is natural to inquire whether the same minimal surface can be defined in more than one way by equations of the form (98). We assume that this is possible, and indicate by u_1, v_1 and $F_1(u_1), \Phi_1(v_1)$ the corresponding parameters and functions. As the parameters u_1, v_1 refer to the lines of length zero on the surface, each is a function of either u or v . In order to determine the forms of the latter we make use of the fact that the positive directions of the normal to the surface in the two forms of parametric representation may have the same or opposite senses. When they have the same sense, the expressions (101) and similar ones in u_1 and v_1 must be equal respectively. In this case

$$(106) \quad u_1 = u, \quad v_1 = v$$

If the senses are opposite, the respective expressions are equal to within algebraic signs. From the resulting equations we find

$$(107) \quad u_1 = -\frac{1}{v}, \quad v_1 = -\frac{1}{u}$$

When we compare equations (98) with analogous equations in u_1 and v_1 , we find that for the case (106) we must have

$$F_1(u_1) = F(u), \quad \Phi_1(v_1) = \Phi(v),$$

and for the case (107)

$$F_1(u_1) = -v^4 \Phi(v), \quad \Phi_1(v_1) = -u^4 F(u)$$

we have the theorem

Necessary and sufficient condition that two minimal surfaces, determined by the pairs of functions F, Φ and F_1, Φ_1 , be congruent is that

$$F_1(u) = -\frac{1}{u^4} \Phi\left(-\frac{1}{u}\right), \quad \Phi_1(v) = -\frac{1}{v^4} F\left(-\frac{1}{v}\right),$$

and the point (u, v) on one surface corresponds the point $\left(-\frac{1}{v}, -\frac{1}{u}\right)$ on the other, and the normals at these points are parallel but of different

direction. In general, the functions F and F_1 as given by (108) are not the same if they are, so also are Φ and Φ_1 . In this case the right-hand sides of equations (98) are unaltered when u and v are replaced by $-1/u$ and $-1/v$ respectively. Hence the Cartesian coordinates of corresponding points (u, v) and $\left(-\frac{1}{v}, -\frac{1}{u}\right)$ differ at most by constants. And regions of the surface about these points either coincide or are brought into coincidence by a translation. In the latter case the surface is periodic and consequently transcendental.

Since it is not periodic, and consider a point $P_0(u_0, v_0)$. As u varies continuously from u_0 to $-1/v_0$, v varies from v_0 to $-1/u_0$, and the point describes a closed curve on the surface by returning to P_0 . But now the positive normal is on the other side of the surface. Hence these surfaces have the property that a point can pass continuously from one side to the other without going through the surface.

On this account they were called *double minimal surfaces*. Schwarz was the first to study them.

In the third theorem of § 110 it follows that double minimal surfaces are characterized by the property that the minimal curves of the two systems are congruent. The equations of such a surface can be written

$$[f_1(u) + f_1(v)], \quad y = \frac{1}{2} [f_2(u) + f_2(v)], \quad z = \frac{1}{2} [f_3(u) + f_3(v)]$$

where the surface is consequently the locus of the mid-points of the chords of the curve

$$\xi = f_1(u), \quad \eta = f_2(u), \quad \zeta = f_3(u),$$

as u varies upon the surface and is the envelope of the parametric

EXAMPLES

1 The focal sheets of a minimal surface are applicable to one another and to the surface of revolution of the evolute of the catenary about the axis of the latter

2 Show that there are no minimal surfaces with the minimal lines in one family straight

3 If two minimal surfaces correspond with parallelism of tangent planes, the minimal curves on the two surfaces correspond

4 If two minimal surfaces correspond with parallelism of tangent planes, and the joins of corresponding points be divided in the same ratio, the locus of the points of division is a minimal surface

5 Show that the right helicoid is defined by $F(u) = im/2u^2$, where m is a real constant, and that it is a double surface

6 The surface for which $F(u) = \frac{2}{1-u^4}$ is called the *surface of Scherk*. Find its equation in the Monge form $z = f(x, y)$. Show that it is doubly periodic and that it is a surface of translation with real generators which are in perpendicular planes

7 By definition a *meridian curve* on a surface is one whose spherical representation is a great circle on the unit sphere. Show that the surface of Scherk possesses two families of plane meridian curves

112 **Algebraic minimal surfaces** Weierstrass* remarked that formulas (98) can be put in a form free of all quadratures. This is done by replacing $F(u)$ and $\Phi(v)$ by $f'''(u)$ and $\phi'''(v)$, where the accents indicate differentiation, and then integrating by parts. This gives

$$(109) \quad \begin{cases} x = \frac{1-u^2}{2} f''(u) + u f'(u) - f(u) + \frac{1-v^2}{2} \phi''(v) + v \phi'(v) - \phi(v), \\ y = i \frac{1+u^2}{2} f''(u) - u f'(u) + f(u) - i \frac{1+v^2}{2} \phi''(v) + v \phi'(v) \\ \quad - i \phi(v), \\ z = u f''(u) - f'(u) + v \phi''(v) - \phi'(v) \end{cases}$$

It is clear that the surface so defined is real when f and ϕ are conjugate imaginary functions. In this case the above formulas may be written

$$(110) \quad \begin{cases} x = R[(1-u^2)f''(u) + 2uf'(u) - 2f(u)], \\ y = Ri[(1+u^2)f''(u) - 2uf'(u) + 2f(u)], \\ z = R[2uf''(u) - 2f'(u)] \end{cases}$$

* *Monatsberichte der Berliner Akademie* (1866), p. 619

r, it is not necessary, as Darboux * has pointed out, that be conjugate imaginaries in order that the surface be real. The relations (109) are unaltered if f and ϕ be replaced by

$$f_1(u) = f(u) + A(1 - u^2) + B_1(1 + u^2) + 2Cu,$$

$$\phi_1(v) = \phi(v) - A(1 - v^2) + B_1(1 + v^2) - 2Cv,$$

where A, B, C are any constants whatever. Evidently, if f and ϕ be conjugate imaginaries, the same is not true in general of f_1 and ϕ_1 , but the surface was real for the former and consequently for the latter also. It is readily found that f_1 and ϕ_1 are conjugate imaginary functions only in case A, B, C are pure imaginaries.

The relations (109) are of particular value in the study of algebraic minimal surfaces.

Thus, it is evident that the surface is algebraic when f and ϕ are algebraic. Conversely, every algebraic minimal surface can be represented by algebraic functions f and ϕ . In proving this we use the method suggested by Weierstrass † and first establish the following lemma.

If a function $\Phi(\xi + i\eta)$ is given and let $\Psi(\xi, \eta)$ denote the real part of Φ in a certain domain, an algebraic relation exists between Ψ and Φ which is an algebraic function of $\xi + i\eta$.

At the point $\xi = 0, \eta = 0$ does not lie within the domain under consideration, this can be effected by a change of variables without affecting the argument. Assuming that this has been done, we expand the function Φ in a power series, thus

$$\Phi = a_0 + ib_0 + (a_1 + ib_1)(\xi + i\eta) + (a_2 + ib_2)(\xi + i\eta)^2 + \dots,$$

where the a 's and b 's are real constants. Evidently Ψ is given by

$$\begin{aligned} \Psi &= a_0 + \frac{1}{2}(a_1 + ib_1)(\xi + i\eta) + \frac{1}{2}(a_1 - ib_1)(\xi - i\eta) + \\ &\quad + \frac{1}{2}(a_2 + ib_2)(\xi + i\eta)^2 + \frac{1}{2}(a_2 - ib_2)(\xi - i\eta)^2 + \dots \end{aligned}$$

Let $\Phi(\xi, \eta) = 0$ denote a rational integral relation between ξ and η . When Ψ has been replaced by the above value, and the expression is arranged in powers of ξ and η , the coefficients of every term are identically zero. They will continue to be zero if ξ and η have been replaced by two complex quantities

α and β , provided that the development remains convergent. The condition for the latter is that the moduli of α and β be each one half the modulus of $\xi + i\eta$. This condition is satisfied if we take

$$\alpha = \frac{\xi + i\eta}{2}, \quad \beta = \frac{\xi + i\eta}{2i}$$

Now we have

$$\Psi(\alpha, \beta) = \frac{1}{2}(a_0 - ib_0) + \frac{1}{2}\Phi(\xi + i\eta),$$

so that

$$F[\Psi(\alpha, \beta), \alpha, \beta] = F_1[\Phi(\xi + i\eta), \xi + i\eta] = 0,$$

which proves the lemma

In applying this lemma to real minimal surfaces we note from (101) that

$$\frac{X}{1-Z} = \frac{u+v}{2}, \quad \frac{Y}{1-Z} = \frac{u-v}{2i},$$

consequently the left-hand members of these equations are equal to u_1 and v_1 respectively, where $u = u_1 + iv_1$. When the surface is algebraic there exists an algebraic relation between the functions

$\frac{X}{1-Z}, \frac{Y}{1-Z}$ and each of the Cartesian coordinates *. Since, then,

there is an algebraic relation between u_1, v_1 , and each of the coordinates given by (110), it follows from the lemma that each of the three expressions

$$\phi_1(u) = (1-u^2)f''(u) + 2uf'(u) - 2f(u),$$

$$\phi_2(u) = i(1+u^2)f''(u) - 2uf'(u) + 2if(u),$$

$$\phi_3(u) = 2uf''(u) - 2f'(u)$$

are algebraic functions of u , and so also is $f(u)$, for,

$$f(u) = \frac{1}{4}(u^2-1)\phi_1(u) - \frac{i}{4}(u^2+1)\phi_2(u) - \frac{1}{2}u\phi_3(u)$$

Hence we have demonstrated the theorem of Weierstrass

The necessary and sufficient condition that equation (110) define an algebraic surface is that $f(u)$ be algebraic

* For if the surface is defined by $F(x, y, z) = 0$, the direction-cosines of the normal are functions of x, y, z . Eliminating two of the latter between $\frac{X}{1-Z}, \frac{Y}{1-Z}$ and $F(x, y, z) = 0$, we have a relation of the kind described

Associate surfaces When the equations of a minimal S are written in the abbreviated form (95), the linear

$$15 \quad ds^2 = 2(dU_1 dV_1 + dU_2 dV_2 + dU_3 dV_3)$$

the linear element also of a surface defined by

$$r_\alpha = e^{i\alpha} U_1 + e^{-i\alpha} V_1, \quad y_\alpha = e^{i\alpha} U_2 + e^{-i\alpha} V_2, \quad z_\alpha = e^{i\alpha} U_3 + e^{-i\alpha} V_3,$$

is any constant. There are an infinity of such surfaces, *associate minimal surfaces*. It is readily found that the directions of the normal to any one have the values (101). Hence associate minimal surfaces defined by (111) have their tangents at corresponding points parallel, and are applicable. Of particular interest is the surface S_1 for which $\alpha = \pi/2$. Its lines are

$$\begin{cases} x_1 = \frac{i}{2} \int (1 - u^2) F(u) du - \frac{i}{2} \int (1 - v^2) \Phi(v) dv, \\ y_1 = -\frac{1}{2} \int (1 + u^2) F(u) du - \frac{1}{2} \int (1 + v^2) \Phi(v) dv, \\ z_1 = i \int u F(u) du - i \int v \Phi(v) dv \end{cases}$$

to show that S_1 is the adjoint (§ 109) of S , we have only that the asymptotic lines on either surface correspond to lines of curvature on the other. For S_1 the equations of the curvature and asymptotic lines are

$$\begin{aligned} iF(u) du^2 + i\Phi(v) dv^2 &= 0, \\ iF(u) du^2 - i\Phi(v) dv^2 &= 0, \end{aligned}$$

respectively. Comparing these with (103) and (104), we see that the real condition is satisfied.

(98) and (112) we obtain the identities

$$\begin{cases} dx + dy' + dz' = dx_1' + dy_1 + dz_1^2, \\ dx dx_1 + dy dy_1 + dz dz_1 = 0 \end{cases}$$

which has the following interpretation

Two adjoint minimal surfaces at points corresponding with parallel tangent planes: the tangents to corresponding curves are perpendicular.

From (105) it follows that if we put

$$\bar{u} + \bar{v} = \int \sqrt{F(u)} du,$$

the curves $\bar{u} = \text{const}$ and $\bar{v} = \text{const}$ on the surface are its lines of curvature. Moreover, for an associate surface the lines of curvature are given by

$$R[e^{\frac{i\alpha}{2}}(\bar{u} + \bar{v})] = \text{const}, \quad R[ie^{\frac{i\alpha}{2}}(\bar{u} + \bar{v})] = \text{const}$$

or

$$\bar{u} \cos \frac{\alpha}{2} - \bar{v} \sin \frac{\alpha}{2} = \text{const}, \quad \bar{u} \sin \frac{\alpha}{2} + \bar{v} \cos \frac{\alpha}{2} = \text{const}$$

From this result follows the theorem

The lines of curvature on a minimal surface associate to a S correspond to the curves on S which cut its lines of curvature the constant angle $\alpha/2$

Since equations (111) may be written

$$(114) \quad \begin{cases} x_\alpha = x \cos \alpha + x_1 \sin \alpha, \\ y_\alpha = y \cos \alpha + y_1 \sin \alpha, \\ z_\alpha = z \cos \alpha + z_1 \sin \alpha, \end{cases}$$

the plane determined by the origin of coordinates, a point on a minimal surface and the corresponding point on its adjoint contains the point P_α corresponding to P on every associate minimal surface. Moreover, the locus of these points P_α is an ellipse with center at the origin. Combining this result and the first part of this section, we have

A minimal surface admits of a continuous deformation into a family of minimal surfaces, and each point of the surface describes an ellipse whose plane passes through a fixed point which is the center of the ellipse

114 Formulas of Schwarz Since the tangent planes to a minimal surface and its adjoint at corresponding points are parallel, we

$$X dx_1 + Y dy_1 + Z dz_1 = 0$$

From this and the second of (113) we obtain the proportion

$$\frac{dx_1}{X dz - Y dz} = \frac{dy_1}{X dz - Z dx} = \frac{dz_1}{Y dx - X dy}$$

sequence of the first of (113) the sums of the squares of the numerators and of the denominators are equal. And so the common value is $+1$ or -1 . If the expressions for the various quantities are substituted from (98), (101), and (112), it is found that the value is -1 . Hence we have

$$dx_1 = Y dz - Z dy, \quad dy_1 = Z dx - X dz, \quad dz_1 = X dy - Y dx$$

these equations and the formulas (95), (112) we have

$$\begin{cases} 2 U_1 = x - x_1 = x + i \int Z dy - Y dz, \\ 2 U_2 = y - y_1 = y + i \int X dz - Z dx, \\ 2 U_3 = z - z_1 = z + i \int Y dx - X dy, \\ 2 V_1 = x + x_1 = x - i \int Z dy - Y dz, \\ 2 V_2 = y + y_1 = y - i \int X dz - Z dx, \\ 2 V_3 = z + z_1 = z - i \int Y dx - X dy \end{cases}$$

These equations are known as the *formulas of Schwarz* *. Their importance is due to their ready applicability to the solution of the problem

To determine a minimal surface passing through a given curve and cutting at each point of the curve a given tangent plane †

In solving this problem we let C be a curve whose coordinates are analytic functions of a parameter t , and let X, Y, Z be functions of t satisfying the conditions

$$X^2 + Y^2 + Z^2 = 1, \quad X dx + Y dy + Z dz = 0$$

*, Vol LXXX (1875) p 291
 This problem is a special case of the more general one solved by Cauchy. *To determine a minimal surface of a differential equation passing through a curve and admitting at each point of the curve a given tangent plane*. For minimal surfaces the equation is $Z = X^2 + Y^2$. He showed that such a surface exists in general and that it is unique unless the curve is characteristic for the equation. His researches are inserted in Vols XIV XV of *Annales des Mathématiques*. The reader may consult also Kowalewski, *Theorie der partiellen Differentialgleichungen* Crelle Vol LXXX (1875), p 1, and Goursat, *Cours d'Analyse* 3^e édition, Vol II, pp 563-567 (Paris, 1905)

If x_u, y_u, z_u denote the values of x, y, z when t is replaced by a complex variable u , and x_v, y_v, z_v the values when t is replaced by v , the equations

$$(118) \quad \begin{cases} \bar{x} = U_1 + V_1 = \frac{x_u + x_v}{2} + \frac{i}{2} \int_v^u (Z dy - Y dz), \\ \bar{y} = U_2 + V_2 = \frac{y_u + y_v}{2} + \frac{i}{2} \int_v^u (X dz - Z dx), \\ \bar{z} = U_3 + V_3 = \frac{z_u + z_v}{2} + \frac{i}{2} \int_v^u (Y dx - X dy) \end{cases}$$

define a minimal surface which passes through C and admits at each point for tangent plane the plane through the point with direction-cosines X, Y, Z . For, when u and v are replaced by t , these equations define C . And the conditions (96) and

$$\sum X dU_1 = 0, \quad \sum X dV_1 = 0$$

are satisfied. Furthermore, the surface defined by (118) affords the unique solution, as is seen from (116) and (117).

When, in particular, C and t are real, the equations of the real minimal surface, satisfying the conditions of the problem, may be put in the form

$$\begin{aligned} \bar{x} &= R \left[x + i \int_t^u (Z dy - Y dz) \right], \\ \bar{y} &= R \left[y + i \int_t^u (X dz - Z dx) \right], \\ \bar{z} &= R \left[z + i \int_t^u (Y dx - X dy) \right] \end{aligned}$$

As an application of these formulas, we consider minimal surfaces containing a straight line. If we take the latter for the z axis, and let ϕ denote the angle which the normal to the surface at a point of the line makes with the x axis, we have

$$x = y = 0, \quad z = t, \quad X = \cos \phi, \quad Y = \sin \phi, \quad Z = 0$$

Hence the equations of the surface are

$$\bar{x} = -Ri \int_t^u \sin \phi dt, \quad \bar{y} = Ri \int_t^u \cos \phi dt, \quad \bar{z} = R(u)$$

Here ϕ is an analytic function of t , whose form determines the character of the surface. For two points corresponding to conjugate values of u , the $\bar{z}$ coordinates are equal, and the $\bar{x}$ - and $\bar{y}$ -coordinates differ in sign. Hence

Every straight line upon a minimal surface is an axis of symmetry

EXAMPLES

tangents to corresponding curves on two associate minimal surfaces meet at constant angle

corresponding directions on two applicable surfaces meet under constant angle if the latter are associate minimal surfaces.

Prove that the catenoid and the right helicoid are adjoint surfaces and determine the function $F(u)$ which defines the former

C be a geodesic on a minimal surface S . Show that the equations of the surface may be put in the form

$$x = R \left[\xi + i \int \lambda ds \right], \quad y = R \left[\eta + i \int \mu ds \right], \quad z = R \left[\zeta + i \int \nu ds \right],$$

ξ, η, ζ are the coordinates of a point on C , and λ, μ, ν the direction-cosines of the tangent to C .

ρ denotes the radius of curvature of C , ρ_1 the radius of first curvature of C , ρ_2 the radius of second curvature of C , π is a plane curve, the surface is symmetric with respect to its plane

surface for which $F(u) = 1 - \frac{1}{u^2}$ is called the *surface of Henneberg*, it is an algebraic surface of the fifteenth order and fifth class.

GENERAL EXAMPLES

edge of regression of the developable surface circumscribed to two conics has for projections on the three principal planes the evolutes of the conics

definition a *tetrahedral surface* is one whose equations are of the form

$$A(u-a)^m(v-a)^n, \quad y = B(u-b)^m(v-b)^n, \quad z = C(u-c)^m(v-c)^n,$$

B, C, m, n are any constants. Show that the parametric curves are conics and that the asymptotic lines can be found by quadratures, also that when the equation of the surface is

$$\frac{x}{A} \frac{1}{(b-c)} + \left(\frac{y}{B} \right)^{\frac{1}{m}} \frac{1}{(c-a)} + \left(\frac{z}{C} \right)^{\frac{1}{m}} \frac{1}{(a-b)} = (a-b)(b-c)(a-c)$$

determine the tetrahedral surfaces, defined as in Ex 2, upon which the principal curves are the lines of curvature

find the surfaces normal to the tangents to a family of umbilical geodesics on a paraboloid, and find the complementary surface

every point of a geodesic circle with center at an umbilical point on the sphere

$$abc = \rho^2 H^2 (a + c - r),$$

ρ is the radius vector of the point (cf § 102)

π tangent plane to the director-cone of a ruled surface along a generator is perpendicular to the tangent plane to the surface at the infinitely distant point on the ruling generator

7 Upon the hyperboloid of one sheet, and likewise upon the hyperbolic paraboloid, the two lines of striction coincide

8 The line of striction of a ruled surface is an orthogonal trajectory of the generators only in case the latter are the binormals of a curve or the surface is a right conoid

9 Determine for a geodesic on a developable surface the relation existing between the curvature, torsion, and angle of inclination of the geodesic with the generators

10 If h denotes the shortest distance and α the angle between two lines l_1 and l_2 , and the latter revolves about the former with a helicoidal motion of parameter a (cf § 62), the locus of l_2 is a developable surface if $a = h \cot \alpha$. If $a = h \tan \alpha$, the surface is the locus of the binormals of a circular helix

11 If the lines of curvature in one family upon a ruled surface are such that the segments of the generators between two curves of the family are of the same length, the parameter of distribution is constant and the line of striction is a line of curvature

12 If two ruled surfaces meet one another in a generator, they are tangent to one another at two points of the generator or at every point, in the latter case the central point for the common generator is the same, and the parameter of distribution has the same value

13 If tangents be drawn to a ruled surface at points of the line of striction and in directions perpendicular to the generators, these tangents form the *conjugate* ruled surface. It has the same line of striction as the given surface. Moreover, a generator of the given surface, the normal to the surface at the central point C of this generator, and the generator of the conjugate surface through C are parallel to the tangent, principal normal, and binormal of a twisted curve

14 Let C be a curve on a surface S , and Σ the ruled surface formed by the normals to S along C . Derive the following results

(a) the distance between near by generators of Σ is of the first order unless C is a line of curvature,

(b) if r denotes the distance from the central point of a generator to the point of intersection with S ,

$$r\Sigma(dX)^2 = -\Sigma dx dX$$

(c) the tangent to C at a point M is conjugate to the tangent to the surface at M parallel to the line of shortest distance,

(d) the maximum and minimum values of r are the principal radii of S , ρ_1 , and ρ_2 , and the above equation may be written $r = \rho_1 \sin^2 \phi + \rho_2 \cos^2 \phi$, where ϕ is the angle which the corresponding line of shortest distance makes with the tangent to the line of curvature corresponding to ρ_2

15 If C and C' are two orthogonal curves on a surface, then at the point of intersection (cf Ex 14)

$$\frac{1}{rR} + \frac{1}{r'R} = \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2}$$

16 If C and C' are two conjugate curves on a surface, then at the point of intersection (cf Ex 14)

$$\frac{1}{r} + \frac{1}{r'} = \frac{1}{\rho_1} + \frac{1}{\rho_2}, \quad \frac{r}{r'} = \frac{R}{R'}$$

two surfaces are applicable, and the radii of first and second curvature geodesic on one surface are equal to the radii of second and first curvature responding geodesic on the other, the surfaces are minimal

a surface for which F in (98) is constant, say 3, is called the *minimal surface*, it possesses the following properties

1. an algebraic surface of the ninth degree whose equation is unaltered if x, y, z are replaced by $y, x, -z$ respectively,

2. it meets the plane $z = 0$ in two orthogonal straight lines,

3. if we put $u = \alpha - i\beta$, the equations of the surface are

$$x = 3\alpha + 3\alpha\beta^2 - \alpha^3, \quad y = 3\beta + 3\alpha^2\beta - \beta^3, \quad z = 3\alpha^2 - 3\beta^2,$$

4. the curves $\alpha = \text{const}$, $\beta = \text{const}$ are the lines of curvature,

5. the lines of curvature are rectifiable unicursal curves of the third order and plane curves, the equations of the planes being

$$x + \alpha z - 3\alpha - 2\alpha^3 = 0, \quad y - \beta z - 3\beta - 2\beta^3 = 0,$$

6. the lines of curvature are represented on the unit sphere by a double family whose planes form two pencils with perpendicular axes which are tangent to the sphere at the same point,

7. the asymptotic lines are twisted cubics,

8. the sections of the surface by the planes $x = 0$ and $y = 0$ are cubics, which are the curves on the surface and the locus of the double points of the lines of curvature

9. the associate minimal surfaces are positions of the original surface rotated through an angle $-\alpha/2$ about the z -axis, where α has the same meaning as in § 113, 10. the surface is the envelope of the plane normal, at the mid point, to the join of the two focal points, one on each of the focal parabolas

$$x = 4\alpha, \quad y = 0, \quad z = 2\alpha^2 - 1, \quad x = 0, \quad y = 4\beta, \quad z = 1 - 2\beta^2,$$

11. the normals to the two parabolas at the extremities of the join are the planes of curvature through the point of contact of the first plane

12. find the equations of Schwarz of a minimal surface when the given curve is an asymptotic line

13. let S and S' be two surfaces, and let the points at which the normals are corresponding for convenience let S and S' be referred to their common centre. Show that if the correspondence is conformal, either S and S' are both spheres, or both are minimal surfaces, or the parametric curves are the lines of curvature on both surfaces, and form an isothermal system

14. find the coordinates of the surface which corresponds to the ellipsoid after the transformation of Ex 20. Show that the surface is periodic, and investigate the points corresponding to the umbilical points on the ellipsoid

15. when the equations of an ellipsoid are in the form (11), the curves $u + v = \text{const}$ are on spheres whose centers coincide with the origin, and at all points of the surface the product ρW is constant (§ 102)

CHAPTER VIII

SURFACES OF CONSTANT TOTAL CURVATURE W-SURFACES SURFACES WITH PLANE OR SPHERICAL LINES OF CURVATURE

115 Spherical surfaces of revolution Surfaces whose total curvature K is the same at all points are called *surfaces of constant curvature*. When this constant value is zero, the surface is developable (§ 64). The nondevelopable surfaces of this kind are called *spherical* or *pseudospherical*, according as K is positive or negative. We consider these two kinds and begin our study of them with the determination of surfaces of revolution of constant curvature.

When upon a surface of revolution the curves $v = \text{const}$ are the meridians and $u = \text{const}$ the parallels, the linear element is reducible to the form

$$(1) \quad ds^2 = du^2 + G dv^2,$$

where G is a function of u alone (§ 46). In this case the expression for the total curvature (V, 12) is

$$(2) \quad K = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial u^2}$$

For spherical surfaces we have $K = 1/a^2$, where a is a real constant. Substituting this value in equation (2) and integrating, we have

$$(3) \quad \sqrt{G} = c \cos \left(\frac{u}{a} + b \right),$$

where b and c are constants of integration. From (1) it is seen that a change in b means simply a different choice of the parallel $u = 0$. If we take $b = 0$, the linear element is

$$(4) \quad ds^2 = du^2 + c^2 \cos^2 \frac{u}{a} dv^2$$

From (III, 99, 100) it follows that the equations of the meridian curve are

$$(5) \quad r = c \cos \frac{u}{a}, \quad z = \int \sqrt{1 - \frac{c^2}{a^2} \sin^2 \frac{u}{a}} du,$$

v measures the angle between the meridian planes. There are three cases to be considered, according as c is equal to, greater than, or less than, a

I $c = a$ Now

$$r = a \cos \frac{u}{a}, \quad z = a \sin \frac{u}{a},$$

evidently the surface is a sphere

II $c > a$ From the expression

it follows that $\sin^2 \frac{u}{a} < 1$ and consequently

$r > 0$ Hence the surface is made up of zones bounded by minimum circles whose radii are equal to the

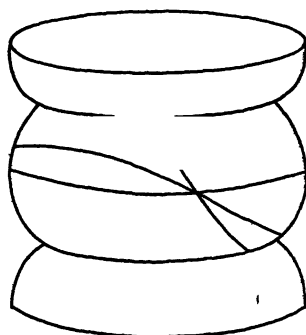


FIG 26

minimum value of $\cos \frac{u}{a}$, and the greatest parallel of each zone is at $u = c$, as in fig 26, where the curves represent geodesics

III $c < a$ Now r varies from 0 to c , the former corresponding to the value $u = m\pi/2$, where m is any odd integer. At these points on the axis the meridians meet the latter under the angle

$\frac{c}{a}$. Hence the surface is made up of a series of spindles

For the cases II and III the expression for z can be integrated in terms of elliptic functions *

It is readily found that these two surfaces are applicable to the sphere with the meridians and parallels of each in correspondence. Thus, if we write the linear element of the sphere in the form

$$ds^2 = d\bar{u}^2 + a^2 \cos^2 \frac{\bar{u}}{a} d\bar{v}^2,$$

it follows from (4) that the equations

$$\bar{u} = u, \quad \bar{v} = \frac{c}{a} v$$

27

determine the correspondence desired

It is evident that for values of b other than zero we should be able to obtain the same results. However, for the sake of future

* Cf. Bianchi, Vol I, p 233

reference we write down the expressions for the linear element when $b = -\pi/2$ and $-\pi/4$ together with (4), thus

$$(6) \quad \begin{cases} (i) & ds^2 = du^2 + c^2 \cos^2 \frac{u}{a} dv^2, \\ (ii) & ds^2 = du^2 + c^2 \sin^2 \frac{u}{a} dv^2, \\ (iii) & ds^2 = du^2 + c^2 \cos^2 \left(\frac{u}{a} - \frac{\pi}{4} \right) dv^2 \end{cases}$$

Let S be a surface with the linear element (6, i), and consider the zone between the parallels $u_0 = \text{const}$ and $u_1 = \text{const}$. A point of the zone is determined by values of u and v such that

$$u_1 \geq u \geq u_0, \quad 2\pi \geq v \geq 0$$

The parametric values of the corresponding point on the sphere are such that

$$u_1 \geq \bar{u} \geq u_0, \quad \frac{2\pi c}{a} \geq \bar{v} \geq 0$$

Hence when $c < a$, the given zone on S does not cover the zone on the sphere between the parallels $u_0 = \text{const}$ and $u_1 = \text{const}$ but when $c > a$ it not only covers it, but there is an overlapping.

116 Pseudospherical surfaces of revolution In order to find the pseudospherical surfaces of revolution we replace K in (2) by $-1/a^2$ and integrate. This gives

$$\sqrt{G} = c_1 \cosh \frac{u}{a} + c_2 \sinh \frac{u}{a},$$

where c_1 and c_2 are constants of integration. We consider first the particular forms of the linear element arising when either of these constants is zero or both are equal. They may be written

$$(7) \quad \begin{cases} (i) & ds^2 = du^2 + c^2 \cosh^2 \frac{u}{a} dv^2, \\ (ii) & ds^2 = du^2 + c^2 \sinh^2 \frac{u}{a} dv^2, \\ (iii) & ds^2 = du^2 + c^2 e^{\frac{2u}{a}} dv^2 \end{cases}$$

Any case other than these may be obtained by taking for $\sqrt{G}$ either of the values $\cosh\left(\frac{u}{a} + b\right)$ or $\sinh\left(\frac{u}{a} + b\right)$, where b is a constant.

ange of the parameter u the corresponding linear elements
cible to (i) or (ii) Hence the forms (7) are the most general.
responding meridian curves are defined by

$$\left\{ \begin{array}{ll} \text{(i)} & r = c \cosh \frac{u}{a}, \quad z = \int \sqrt{1 - \frac{c^2}{a^2} \sinh^2 \frac{u}{a}} du, \\ \text{(ii)} & r = c \sinh \frac{u}{a}, \quad z = \int \sqrt{1 - \frac{c^2}{a^2} \cosh^2 \frac{u}{a}} du, \\ \text{(iii)} & r = ce^{\frac{u}{a}}, \quad z = \int \sqrt{1 - \frac{c^2}{a^2} e^{\frac{2u}{a}}} du. \end{array} \right.$$

sider these three cases in detail

I The maximum and minimum values of $\sinh^2 \frac{u}{a}$ are a^2/c^2 .
Hence the maximum and minimum values of r are $\sqrt{a^2 + c^2}$.
At points of a maximum parallel the tangents to the merid-
perpendicular to the axis, and at
of a minimum parallel they are par-
the axis. Hence the former is a cus-
ge, and the latter a circle of gorge,
the surface is made up of spool-like

It is represented by fig 28, upon
he closed curves are geodesic circles
other curves are geodesics. These
pherical surfaces are said to be of
erboloc type *

II In order that the surface be real
ot be greater than a' , a restriction
essary in either of the other cases
ut $c = a \sin \alpha$,† the maximum and

m values of $\cosh^2 \frac{u}{a}$ are $\operatorname{cosec}^2 \alpha$ and 1, and the correspond-
ies of r are $a \cos \alpha$ and 0. The tangents to the meridians
s of the former circle are perpendicular to the axis, and at
its for which r is zero they meet the axis under the angle α .
the surface is made up of a series of parts similar in shape

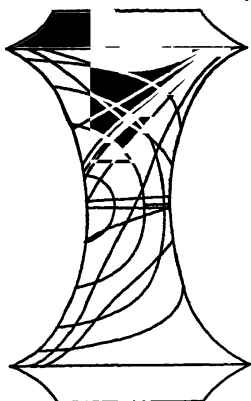


FIG 28

* Cf Bianchi, Vol I, p 223

† Cf Bianchi, Vol I p 226

to hour-glasses Fig 29 represents one half of such a part, one of the curves is an asymptotic line and the others are parallel geodesics. The surface is called a *pseudospherical surface of the elliptic type*

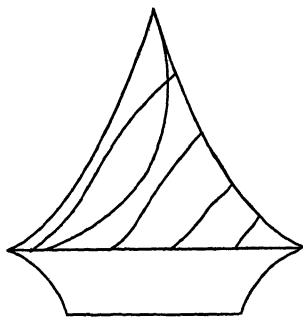


FIG 29

CASE III In the preceding cases the equations of the meridian curve can be expressed without the quadrature sign by means of elliptic functions. In this case the same can be done by means of trigonometric functions. For if we put

$$\sin \phi = \frac{c}{a} e^{\frac{u}{a}},$$

equations (11) of (8) become

$$(9) \quad r = a \sin \phi, \quad z = a \left(\log \tan \frac{\phi}{2} + \cos \phi \right)$$

We find that ϕ is the angle which the tangent to a meridian at point makes with the axis. Hence the axis is an asymptote to the curve. Since the length of the segment of a tangent between the point of contact and the intersection with the axis is $r \operatorname{cosec} \phi$ or a , the length of the segment is independent of the point of contact. Therefore the meridian curve is a *tractrix*. The surface of revolution of a tractrix about its asymptote is called the *pseudosphere*, or the *pseudospherical surface of the parabolic type*. The surface is shown in fig 30, which also pictures a family of parallel geodesics and an asymptotic line.

If the integral (3) be written in the form

$$\sqrt{G} = c_1 \cos \frac{u}{a} + c_2 \sin \frac{u}{a},$$

the cases (1), (11), (111) of (6) are seen to correspond to the similar cases of (7). We shall find other marks of similarity between these cases, but now we desire to call attention to differences.

Each of the three forms (7) determines a particular kind of pseudospherical surface of revolution, and c is restricted in value

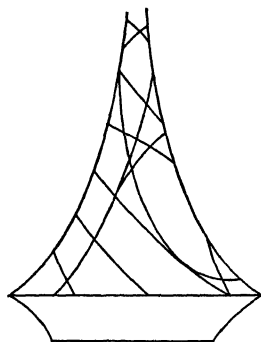


FIG 30

the second case. On the contrary each of the three forms is to define any of the three types of spherical surfaces of revolution according to the magnitude of c .

(IV, 51) we find that the geodesic curvature of the parallel surfaces with the linear elements (7) is measured by the expressions

$$\frac{1}{a} \tanh \frac{u}{a}, \quad \frac{1}{a} \coth \frac{u}{a}, \quad \frac{1}{a}$$

Two of these expressions can be transformed into the other by replacing u by u plus any constant, it follows that two spherical surfaces of revolution of different types are not transformable into one another with meridians in correspondence.

Geodesic parametric systems. Applicability. Now we shall see that in corresponding cases of (6) and (7) the parametric systems are of the same kind, and then we shall prove that when such a geodesic system is chosen for any surface of constant curvature, not necessarily one of revolution, the metric element can be brought to the corresponding form of (6).

In the first place we recall that when on any surface the curves $u = \text{const}$ are geodesics, and $u = \text{const}$ their orthogonal trajectories, a line element is reducible to the form (1), where G is, in general, a function of both u and v , and the geodesic curvature of the curves $u = \text{const}$ is given by (IV, 51), namely

$$\frac{1}{\rho_g} = \frac{1}{\sqrt{EG}} \frac{\partial \sqrt{G}}{\partial u}$$

In particular, the curvature of the surface is constant, $\sqrt{G}$ is given by equation (2) in which K may be replaced by $\pm 1/a^2$. For spherical surfaces, the general form of $\sqrt{G}$ is

$$\sqrt{G} = \phi(v) \cos \frac{u}{a} + \psi(v) \sin \frac{u}{a},$$

for pseudospherical surfaces

$$\sqrt{G} = \phi(v) \cosh \frac{u}{a} + \psi(v) \sinh \frac{u}{a},$$

where ϕ and ψ are, at most, functions of v . We consider now the cases of (6) and (7)

CASE I From the forms (i) of (6) and (7), and from (10), it follows that the curve $u=0$ is a geodesic and that its arc is measured by cv . Moreover, a necessary and sufficient condition that the curve $u=0$ on any surface with the linear element (I) satisfy these conditions is

$$\left(\sqrt{G}\right)_{u=0} = c, \quad \left(\frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial u}\right)_{u=0} = 0$$

Applying these conditions to (11) and (12), we are brought to the forms (i) of (6) and (7) respectively

CASE II The forms (ii) of (6) and (7) satisfy the conditions

$$\left(\sqrt{G}\right)_{u=0} = 0, \quad \left(\frac{\partial}{\partial u} \sqrt{G}\right)_{u=0} = c,$$

which are necessary and sufficient that the parametric system be geodesic polar, in which cv measures angles (cf VI, 54). When these conditions are applied to (11) and (12), we obtain (ii) of (6) and of (7) respectively

CASE III For (iii) of (6) the curve $u=0$ has constant geodesic curvature $1/a$, and for (iii) of (7) all of the curves $u=\text{const}$ have the same geodesic curvature $1/a$. Conversely, we find from (11) and (12) that when this condition is satisfied on any surface of constant curvature the linear element is reducible to one of the forms (iii). We gather these results together into the theorem

The linear element of any surface of constant curvature is reducible to the forms (i), (ii), (iii) of (6) or (7) according as the parametric geodesics are orthogonal to a geodesic, pass through a point, or are orthogonal to a curve of constant geodesic curvature

When the linear element of a surface of constant curvature is in one of the forms (i), (ii), (iii) of (6) and (7), it is said to be of the *hyperbolic, elliptic, or parabolic* type accordingly.

The above theorem may be stated as follows

Any spherical surface of curvature $1/a^2$ is applicable to a sphere of radius a in such a way that to a family of great circles with the same diameter there correspond the geodesics orthogonal to a

geodesic on the surface, or all the geodesics through any point, or those which are orthogonal to a curve of geodesic curvature $1/a$

pseudospherical surface of curvature $-1/a^2$ is applicable to a spherical surface of revolution of any of the three types, and as the latter surface is of the hyperbolic, elliptic, or parabolic type, to its meridians correspond on the given surface geodesics which are orthogonal to a geodesic, or pass through a point, or are tangent to a curve of geodesic curvature $1/a$.

In the case of spherical surfaces one system of geodesics can satisfy all three conditions, for in the case of the sphere the great circles with the same diameter are orthogonal to the equator, pass through both poles, and are orthogonal to two small circles of radius $1/\sqrt{2}$, whose geodesic curvature is $1/a$. But on a pseudospherical surface a geodesic system can satisfy only one of these conditions. Otherwise it would be possible to apply two surfaces of different types in such a way that meridians and parallels correspond.

From the foregoing theorems it follows that, in order to carry out the applicability of a surface of constant curvature upon any surface of revolution, it is only necessary to find the geodesics on the given surface. The nature of this problem is indicated in the theorem

Determination of the geodesic lines on a surface of constant curvature requires the solution of a Riccati equation

Proving this theorem we consider first a spherical surface in terms of any parametric system. It is applicable to a sphere of the same curvature with center at the origin. The coordinates of this sphere, expressed as functions of the parameters u, v , can be found by the solution of a Riccati equation (65). To great circles on the sphere correspond geodesics on the spherical surface, hence the finite equation of geodesics is $a\bar{x} + b\bar{y} + c\bar{z} = 0$, where a, b, c are arbitrary constants.

If the surface is pseudospherical we use an imaginary sphere of the same curvature, and the analysis is similar.

tion of Hazzidakis Let a spherical surface be defined in terms of isothermal-conjugate param-

$$\frac{D}{H} = \frac{D''}{H} = \frac{1}{a},$$

the Codazzi equations (V, 13') reduce to

$$(14) \quad \begin{cases} \frac{\partial E}{\partial u} - \frac{\partial G}{\partial u} + 2 \frac{\partial F}{\partial v} = 0, \\ \frac{\partial E}{\partial v} - \frac{\partial G}{\partial v} - 2 \frac{\partial F}{\partial u} = 0 \end{cases}$$

From these equations follows the theorem

The lines of curvature of a spherical surface form an isothermal conjugate system

For, a solution of these equations is

$$E - G = \text{const}, \quad F = 0$$

When this constant is zero the surface is a sphere because of (13) Excluding this case, we replace the above by

$$(15) \quad E = a^2 \cosh^2 \omega, \quad F = 0, \quad G = a^2 \sinh^2 \omega$$

Now

$$(16) \quad D = D'' = a \sinh \omega \cosh \omega$$

When these values are substituted in the Gauss equation (V, 12) namely

$$(17) \quad \frac{1}{2H} \left\{ \frac{\partial}{\partial v} \left[\frac{2}{H} \frac{\partial F}{\partial u} - \frac{1}{H} \frac{\partial E}{\partial v} - \frac{F}{H^2} \frac{\partial E}{\partial u} \right] + \frac{\partial}{\partial u} \left[\frac{F}{H^2} \frac{\partial E}{\partial v} - \frac{1}{H} \frac{\partial G}{\partial u} \right] \right\} = \frac{1}{a}$$

it is found that ω must satisfy the equation

$$(18) \quad \frac{\partial^2 \omega}{\partial u^2} + \frac{\partial^2 \omega}{\partial v^2} + \sinh \omega \cosh \omega = 0$$

Conversely, for each solution of this equation the quantities (15) and (16) determine a spherical surface

If equations (14) be differentiated with respect to u and v respectively, and the resulting equations be added, we have

$$(19) \quad \frac{\partial^2 E}{\partial u^2} + \frac{\partial^2 E}{\partial v^2} = \frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2}$$

* The ambiguity of sign may be neglected as a change of sign gives a surface symmetrical with respect to the origin

sequence of (14) equation (17) is reducible to

$$\frac{1}{4H^4} \left\{ E \left[\left(\frac{\partial G}{\partial u} \right)^2 + \left(\frac{\partial G}{\partial v} \right)^2 \right] + 2F \left[\frac{\partial E}{\partial u} \frac{\partial G}{\partial v} - \frac{\partial E}{\partial v} \frac{\partial G}{\partial u} \right] \right. \\ \left. + G \left[\left(\frac{\partial E}{\partial u} \right)^2 + \left(\frac{\partial E}{\partial v} \right)^2 \right] \right\} - \frac{1}{2H^2} \left(\frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2} \right) = \frac{1}{a^2}$$

as (14) are unaltered if E and G be interchanged and the F be changed. The same is true of (17) because of (19).

Hence we have

linear element of a spherical surface referred to an isothermal system of parameters be

$$ds^2 = E du^2 + 2F dudv + G dv^2,$$

exists a second spherical surface of the same curvature referred to a similar parametric system with the linear element

$$ds_1^2 = G du^2 - 2F dudv + E dv^2,$$

with the same second quadratic form as the given surface, then, the lines of curvature correspond on the two surfaces

This fact is evident from the equation of the lines of curvature (7, 26), which reduces to $F du^2 + (G - E) dudv - F dv^2 = 0$

(V, 69) it is seen that the linear elements of the spherical representation of the respective surfaces are

$$d\sigma^2 = \frac{1}{a^2} (G du^2 - 2F dudv + E dv^2),$$

$$d\sigma_1^2 = \frac{1}{a^2} (E du^2 + 2F dudv + G dv^2)$$

Consequently we have the theorem

A solution ω of equation (18) determines two spherical surfaces of curvature $1/a^2$, the linear elements of the surfaces are

$$ds^2 = a^2 (\cosh' \omega du' + \sinh' \omega dv'),$$

$$ds_1^2 = a^2 (\sinh \omega du' + \cosh' \omega dv'),$$

then spherical representations

$$\begin{cases} d\sigma^2 = \sinh \omega du' + \cosh \omega dv', \\ d\sigma_1^2 = \cosh' \omega du' + \sinh' \omega dv', \end{cases}$$

then, the principal radii are respectively

$$\begin{aligned} \rho_1 &= a \coth \omega, & \rho_2 &= a \tanh \omega, \\ \rho'_1 &= a \tanh \omega, & \rho'_2 &= a \coth \omega \end{aligned}$$

Bianchi* has given the name *Hazzidakis transformation* to the relation between these two surfaces. It is evident that the former theorem defines this transformation in a more general way.

119 Transformation of Bianchi We consider now a pseudospherical surface of curvature $-1/a^2$, defined in terms of isothermal conjugate parameters. We have

$$\frac{D}{H} = -\frac{D''}{H} = -\frac{1}{a}, \dagger$$

and the Codazzi equations reduce to

$$\frac{\partial E}{\partial u} + \frac{\partial G}{\partial u} - 2 \frac{\partial F}{\partial v} = 0, \quad \frac{\partial E}{\partial v} + \frac{\partial G}{\partial v} - 2 \frac{\partial F}{\partial u} = 0$$

These equations are satisfied by the values

$$(22) \quad E = a^2 \cos^2 \omega, \quad F = 0, \quad G = a^2 \sin^2 \omega,$$

where ω is a function which, because of the Gauss equation (V, 12) must satisfy the equation

$$(23) \quad \frac{\partial^2 \omega}{\partial u^2} - \frac{\partial^2 \omega}{\partial v^2} = \sin \omega \cos \omega$$

Conversely, every solution of this equation determines a pseudospherical surface whose fundamental quantities are given by (22) and by

$$(24) \quad D = -D'' = -a \sin \omega \cos \omega$$

Moreover, the linear element of the spherical representation is

$$(25) \quad d\sigma^2 = \sin^2 \omega du^2 + \cos^2 \omega dv^2$$

There is not a transformation for pseudospherical surfaces similar to the Hazzidakis transformation of spherical surfaces, but there are transformations of other kinds which are of great importance. One of these is involved in the following theorem of Ribaucour:

If in the tangent planes to a pseudospherical surface of curvature $-1/a^2$ circles of radius a be described with centers at the points of contact, these circles are the orthogonal trajectories of an infinity of surfaces of curvature $-1/a^2$

* Vol II p 437

† This choice of sign is made so that the following formulas may have the customary form

Using this theorem we imagine the given surface S referred to axes of curvature, and we associate with it the moving trihedron whose axes are tangent to the parametric lines. From (22) (75, 76) it follows that

$$\begin{aligned} &= 0, \quad p_1 = \cos \omega, \quad q = \sin \omega, \quad q_1 = 0, \quad r = \frac{\partial \omega}{\partial v}, \quad r_1 = \frac{\partial \omega}{\partial u}, \\ &\xi = a \cos \omega, \quad \eta_1 = a \sin \omega, \quad \xi_1 = \eta = 0 \end{aligned}$$

In the tangent xy -plane we draw from the origin M a segment of length a , and let θ denote its angle of inclination with the x -axis. The coordinates of the other extremity M_1 with respect to these axes are $a \cos \theta$, $a \sin \theta$, 0, and the projections upon these axes of the displacement of M_1 as M moves over S are, by (V, 51),

$$\begin{aligned} &a \left[-\sin \theta \, d\theta + \cos \omega \, du - \left(\frac{\partial \omega}{\partial v} du + \frac{\partial \omega}{\partial u} dv \right) \sin \theta \right], \\ &a \left[\cos \theta \, d\theta + \sin \omega \, dv + \left(\frac{\partial \omega}{\partial v} du + \frac{\partial \omega}{\partial u} dv \right) \cos \theta \right], \\ &a [\cos \omega \sin \theta \, dv - \sin \omega \cos \theta \, du] \end{aligned}$$

now the conditions which θ must satisfy in order that the line l_1 be tangent to the locus of M_1 denoted by S_1 , and that the plane to S_1 at M_1 be perpendicular to the tangent plane to S . Under these conditions the direction-cosines of the tangent to S_1 with reference to the moving trihedral are

$$\sin \theta, \quad -\cos \theta, \quad 0,$$

so the tangent to the above displacement must be in this direction. We have

$$d\theta + \left(\frac{\partial \omega}{\partial v} - \sin \theta \cos \omega \right) du + \left(\frac{\partial \omega}{\partial u} + \cos \theta \sin \omega \right) dv = 0$$

For this equation must hold for all displacements of M , it is equivalent to

$$\begin{cases} \frac{\partial \theta}{\partial u} + \frac{\partial \omega}{\partial v} = \cos \omega \sin \theta, \\ \frac{\partial \theta}{\partial v} + \frac{\partial \omega}{\partial u} = -\sin \omega \cos \theta \end{cases}$$

These equations satisfy the condition of integrability in consequence of (23). Moreover, θ is a solution of equation (23), as is seen by differentiating equations (28) with respect to u and v respectively and subtracting

By means of (28) the above expressions for the projections of displacement of M_1 can be put in the form

$$\begin{aligned} a \cos \theta (\cos \omega \cos \theta du + \sin \omega \sin \theta dv), \\ a \sin \theta (\cos \omega \cos \theta du + \sin \omega \sin \theta dv), \\ a (\cos \omega \sin \theta dv - \sin \omega \cos \theta du) \end{aligned}$$

From these it follows that the linear element of S_1 is

$$ds_1^2 = a^2 (\cos^2 \theta du^2 + \sin^2 \theta dv^2)$$

In order to prove that S_1 is a pseudospherical surface referred to its lines of curvature, it remains for us to show that the spherical representation of these curves forms an orthogonal system. We obtain this representation with the aid of a trihedral whose vertex is fixed, and which rotates so that its axes are always parallel to the corresponding axes of the trihedral for S . The point whose coordinates with reference to the new trihedral are given by (26) serves for the spherical representation of S_1 . The projections upon these axes of a displacement of this point are reducible, by means of (28), to

$$\begin{aligned} \cos \theta (\cos \omega \sin \theta du - \sin \omega \cos \theta dv), \\ \sin \theta (\cos \omega \sin \theta du - \sin \omega \cos \theta dv), \\ - \sin \omega \sin \theta du - \cos \omega \cos \theta dv, \end{aligned}$$

from which it follows that the linear element is

$$d\sigma_1^2 = \sin^2 \theta du^2 + \cos^2 \theta dv^2$$

Since θ is a solution of (23), the surface S_1 is pseudospherical, of curvature $-1/a^2$, and the lines of curvature are parametric. To each solution θ of equations (28) there corresponds a surface S_1 . Darboux* has called this process of finding S_1 the *transformation of Bianchi*. As the complete integral of equations (28) involves an arbitrary constant, there are an infinity of surfaces S_1 , as remarked by Ribaucour. Moreover, if we put

$$(29) \quad \phi = \tan \frac{\theta}{2},$$

these equations are of the Riccati type in ϕ . Hence, by § 14,

When one transform of Bianchi of a pseudospherical surface is known, the determination of the others requires only quadratures

[, 24) it follows that the differential equation of the high the lines joining corresponding points on S and ent is

$$\cos \omega \sin \theta du - \sin \omega \cos \theta dv = 0$$

g such a curve, equation (27) reduces to

$$d\theta + \frac{\partial \omega}{\partial v} du + \frac{\partial \omega}{\partial u} dv = 0$$

VI, 56) it is seen that this is the Gauss equation of pon a surface whose first fundamental coefficients lues (22) Hence

s on S to which the lines joining corresponding points on ; tangent are geodesics

nal trajectories of the curves (30) are defined by

$$\cos \omega \cos \theta du + \sin \omega \sin \theta dv = 0$$

of (28) the left-hand member of this equation is an exact differential

$$d\xi = -a(\cos \omega \cos \theta du + \sin \omega \sin \theta dv),$$

ξ/a is an integrating factor of the left-hand member of (30) Conse-
y define a function η thus

$$d\eta = ae^{-\xi/a}(\cos \omega \sin \theta du - \sin \omega \cos \theta dv)$$

nd η the linear element of S is expressible in the parabolic form (7),

$$ds^2 = d\xi^2 + e^{2\xi/a} d\eta^2$$

1) defines also the orthogonal trajectories of the curves on S_1 to s MM_1 are tangent, and the equation of the latter curves is

$$\sin \omega \cos \theta du - \cos \omega \sin \theta dv = 0$$

$e^{\xi/a}$ is an integrating factor of this equation, and if we put accordingly

$$d\xi = ae^{\xi/a}(\sin \omega \cos \theta du - \cos \omega \sin \theta dv),$$

ent of S_1 may be expressed in the parabolic form

$$ds_1^2 = d\xi^2 + e^{-2\xi/a} d\xi^2$$

ssions (32) and (33) are of the form of the linear element of a surface of finite equations of the geodesics can be found by quadratures Hence
nchi transformation is known for a surface the finite equation of its
e found by quadratures

also from the preceding theorem and the last one of § 117

120. Transformation of Backlund The transformation of Bianchi is only a particular case of a transformation discovered by Backlund, by means of which from one pseudospherical surface S another S_1 of the same curvature, can be found. Moreover, on these two surfaces the lines of curvature correspond, the join of corresponding points is tangent at these points to the surfaces and is of constant length, and the tangent planes at corresponding points meet under constant angle.

We refer S to the same moving trihedral as in the preceding case, and let λ and θ denote the length of MM_1 and the angle which the latter makes with the x -axis. The coordinates of M_1 are $\lambda \cos \theta$, $\lambda \sin \theta$, 0, and the projections of a displacement of M_1 are

$$(34) \quad \begin{cases} -\lambda \sin \theta d\theta + a \cos \omega du - \lambda \sin \theta \left(\frac{\partial \omega}{\partial v} du + \frac{\partial \omega}{\partial u} dv \right), \\ \lambda \cos \theta d\theta + a \sin \omega dv + \lambda \cos \theta \left(\frac{\partial \omega}{\partial v} du + \frac{\partial \omega}{\partial u} dv \right), \\ \lambda (\cos \omega \sin \theta dv - \sin \omega \cos \theta du) \end{cases}$$

If σ denotes the constant angle between the tangent planes to S and S_1 at M and M_1 respectively, since these planes are to intersect in MM_1 , the direction-cosines of the normal to S_1 are

$$\sin \sigma \sin \theta, \quad -\sin \sigma \cos \theta, \quad \cos \sigma$$

Hence θ must satisfy the condition

$$\begin{aligned} \lambda \sin \sigma d\theta - a \sin \sigma (\cos \omega \sin \theta du - \sin \omega \cos \theta dv) \\ + \lambda \sin \sigma \left(\frac{\partial \omega}{\partial v} du + \frac{\partial \omega}{\partial u} dv \right) \\ + \lambda \cos \sigma (\sin \omega \cos \theta du - \cos \omega \sin \theta dv) = 0 \end{aligned}$$

Since this condition must be satisfied for every displacement, it is equivalent to

$$\begin{aligned} \lambda \sin \sigma \left(\frac{\partial \theta}{\partial u} + \frac{\partial \omega}{\partial v} \right) &= a \sin \sigma \cos \omega \sin \theta - \lambda \cos \sigma \sin \omega \cos \theta, \\ \lambda \sin \sigma \left(\frac{\partial \theta}{\partial v} + \frac{\partial \omega}{\partial u} \right) &= -a \sin \sigma \sin \omega \cos \theta + \lambda \cos \sigma \cos \omega \sin \theta \end{aligned}$$

*Om ytor med konstant negativ krokning, *Lunds Universitets Arsskrift* Vol. XIX (1883). An English translation of this memoir has been made by Miss Emily Coddington of New York, and privately printed.

equations be differentiated with respect to v and u respectively and the resulting equations be subtracted, we have

$$a^2 \sin^2 \sigma - \lambda^2 = 0,$$

such it follows that λ is a constant. Without loss of generality we take $\lambda = a \sin \sigma$. If this value be substituted in the equations, we have

$$\begin{cases} \sin \sigma \left(\frac{\partial \theta}{\partial u} + \frac{\partial \omega}{\partial v} \right) = \sin \theta \cos \omega - \cos \sigma \cos \theta \sin \omega, \\ \sin \sigma \left(\frac{\partial \theta}{\partial v} + \frac{\partial \omega}{\partial u} \right) = -\cos \theta \sin \omega + \cos \sigma \sin \theta \cos \omega, \end{cases}$$

the equations satisfy the condition of integrability. If they be differentiated with respect to u and v respectively, and the resulting equations be subtracted, it is found that θ is a solution

in consequence of (35) the expressions (34) reduce to

$$\begin{aligned} & \theta (\cos \omega \cos \theta + \cos \sigma \sin \omega \sin \theta) du \\ & \quad + a \sin \theta (\sin \omega \cos \theta - \cos \sigma \cos \omega \sin \theta) dv, \\ & \theta (\cos \omega \sin \theta - \cos \sigma \sin \omega \cos \theta) du \\ & \quad + a \sin \theta (\sin \omega \sin \theta + \cos \sigma \cos \omega \cos \theta) dv, \\ & \sigma (\cos \omega \sin \theta dv - \sin \omega \cos \theta du), \end{aligned}$$

linear element of S_1 is

$$ds_1^2 = a^2 (\cos^2 \theta du^2 + \sin^2 \theta dv^2)$$

inner similar to that of § 119 it can be shown that the linear representation of the parametric curves is orthogonal, consequently these curves are the lines of curvature on S_1 . The equations (35) are reducible to the Riccati form by the change of variable (29). Moreover, the general solution of these equations involves two constants, namely σ and the constant of integration. We have the theorem

From the integration of a Riccati equation a double infinity of pseudospherical surfaces can be obtained from a given surface of this kind

and we return to this as the *transformation of Backlund*, and indicate the method, thus putting in evidence the constant σ

121 Theorem of permutability Let S_1 be a transform of S by means of the functions (θ_1, σ_1) . Since conversely S is a transform of S_1 and the equations for the latter similar to (35) are reducible to the same type, all the transforms of S_1 can be found by quadratures. Even these quadratures can be dispensed with because of the following theorem of permutability due to Bianchi*:

S_1 and S_2 are transforms of S by means of the respective pairs of functions (θ_1, σ_1) and (θ_2, σ_2) , a function ϕ can be found without quadratures which is such that by means of the pairs (ϕ, σ_2) and (ϕ, σ_1) the surfaces S_1 and S_2 respectively are transformable into a pseudospherical surface S'

By hypothesis ϕ is a solution of the equations

$$(36) \quad \begin{cases} \sin \sigma_2 \left(\frac{\partial \phi}{\partial u} + \frac{\partial \theta_1}{\partial v} \right) = \sin \phi \cos \theta_1 - \cos \sigma_2 \cos \phi \sin \theta_1, \\ \sin \sigma_2 \left(\frac{\partial \phi}{\partial v} + \frac{\partial \theta_1}{\partial u} \right) = -\cos \phi \sin \theta_1 + \cos \sigma_2 \sin \phi \cos \theta_1, \end{cases}$$

and also of the equations

$$(37) \quad \begin{cases} \sin \sigma_1 \left(\frac{\partial \phi}{\partial u} + \frac{\partial \theta_2}{\partial v} \right) = \sin \phi \cos \theta_2 - \cos \sigma_1 \cos \phi \sin \theta_2, \\ \sin \sigma_1 \left(\frac{\partial \phi}{\partial v} + \frac{\partial \theta_2}{\partial u} \right) = -\cos \phi \sin \theta_2 + \cos \sigma_1 \sin \phi \cos \theta_2 \end{cases}$$

The projections of the line M_1M' on the tangents to the lines of curvature of S_1 and on its normal, where M_1 and M' are corresponding points on S_1 and S' , are

$$(38) \quad a \sin \sigma_2 \cos \phi, \quad a \sin \sigma_2 \sin \phi, \quad 0$$

The direction-cosines of the tangents to the lines of curvature of S_1 with respect to the line MM_1 , the line MQ_1 perpendicular to the latter and in the tangent plane at M , and the normal to S are

$$\begin{array}{lll} \cos \omega, & -\cos \sigma_1 \sin \omega, & -\sin \sigma_1 \sin \omega, \\ \sin \omega, & \cos \sigma_1 \cos \omega, & \sin \sigma_1 \cos \omega \end{array}$$

From these and (38) it follows that the coordinates of M' with respect to MM_1 , MQ_1 , and the normal to S are

$$\begin{aligned} a[\sin \sigma_1 + \sin \sigma_2 \cos(\phi - \omega)], & \quad a[\sin \sigma_2 \cos \sigma_1 \sin(\phi - \omega)], \\ a[\sin \sigma_1 \sin \sigma_2 \sin(\phi - \omega)] \end{aligned}$$

coordinates of M' with respect to the axes of the moving for S are

$$\begin{cases} \frac{x'}{a} = \cos \theta_1 \sin \sigma_1 + \cos \theta_1 \sin \sigma_2 \cos(\phi - \omega) \\ \quad - \sin \theta_1 \sin \sigma_2 \cos \sigma_1 \sin(\phi - \omega), \\ \frac{y'}{a} = \sin \theta_1 \sin \sigma_1 + \sin \theta_1 \sin \sigma_2 \cos(\phi - \omega) \\ \quad + \cos \theta_1 \sin \sigma_2 \cos \sigma_1 \sin(\phi - \omega), \\ \frac{z'}{a} = \sin \sigma_1 \sin \sigma_2 \sin(\phi - \omega) \end{cases}$$

are transformed by means of σ_1 and the same function ϕ , coordinates x'', y'', z'' of the resulting surface can be obtained by interchanging the subscripts 1 and 2. Evidently z' is equal to z'' . A necessary and sufficient condition that x', y' be equal to x'', y'' respectively is

$$\begin{aligned} \cos \theta_1(x' - x'') + \sin \theta_1(y' - y'') &= 0, \\ \cos \theta_2(x' - x'') + \sin \theta_2(y' - y'') &= 0 \end{aligned}$$

When these values be substituted in these equations, we obtain

$$\begin{aligned} &[\cos(\theta_2 - \theta_1) - \sin \sigma_2] \cos(\phi - \omega) \\ &+ \sin \sigma_1 \cos \sigma_2 \sin(\theta_2 - \theta_1) \sin(\phi - \omega) = \sin \sigma_1 - \sin \sigma_2 \cos(\theta_2 - \theta_1), \\ &[\cos(\theta_2 - \theta_1) - \sin \sigma_1] \cos(\phi - \omega) \\ &+ \sin \sigma_2 \cos \sigma_1 \sin(\theta_2 - \theta_1) \sin(\phi - \omega) = \sin \sigma_2 - \sin \sigma_1 \cos(\theta_2 - \theta_1) \end{aligned}$$

These equations with respect to $\sin(\phi - \omega)$ and $\cos(\phi - \omega)$,

$$\begin{aligned} \cos(\phi - \omega) &= \frac{(\cos \sigma_2 - \cos \sigma_1) \sin(\theta_2 - \theta_1)}{\sin \sigma_1 \sin \sigma_2 \cos(\theta_2 - \theta_1) + \cos \sigma_1 \cos \sigma_2 - 1}, \\ \sin(\phi - \omega) &= \frac{\sin \sigma_1 \sin \sigma_2 + (\cos \sigma_1 \cos \sigma_2 - 1) \cos(\theta_2 - \theta_1)}{\sin \sigma_1 \sin \sigma_2 \cos(\theta_2 - \theta_1) + \cos \sigma_1 \cos \sigma_2 - 1} \end{aligned}$$

These expressions satisfy the condition that the sum of their squares be unity, and the function ϕ satisfies equations (36) and (37). Hence our hypotheses are consistent and the theorem of permutability is demonstrated.

We may replace the above equations by

$$\tan\left(\frac{\phi - \omega}{2}\right) = \frac{\sin\left(\frac{\sigma_1 + \sigma_2}{2}\right)}{\sin\left(\frac{\sigma_1 - \sigma_2}{2}\right)} \tan\left(\frac{\theta_1 - \theta_2}{2}\right)$$

The preceding result may be expressed in the following form:

When the transforms of a given pseudospherical surface are known, all the transformations of the former can be effected by algebraic processes and differentiation

Thus, suppose* that the complete integral of equations (35) is

$$(41) \quad \theta = f(u, v, \sigma, c),$$

and that a particular integral is

$$\theta_1 = f(u, v, \sigma_1, c_1),$$

corresponding to particular values of the constants, and let S_1 denote the transform of S by means of θ_1 and σ_1 . All the transformations of S_1 are determined by the functions ϕ and σ , where

$$(42) \quad \tan\left(\frac{\phi - \omega}{2}\right) = \frac{\sin\left(\frac{\sigma + \sigma_1}{2}\right)}{\sin\left(\frac{\sigma - \sigma_1}{2}\right)} \tan\left(\frac{f - \theta_1}{2}\right)$$

Exceptional cases arise when σ has the value σ_1 . For all values of c other than c_1 formula (42) gives $\phi = \omega + m\pi$, where m is an odd integer. When this is substituted in equations (36) they reduce to (35). In this case S' coincides with S .

We consider now the remaining case where c has the value c_1 , whereupon the right-hand member of (42) is indeterminate. In order to handle this case we consider c in (41) to be a function of σ , reducing to c_1 for $\sigma = \sigma_1$. If we apply the ordinary methods to the function $\tan\left(\frac{f - \theta_1}{2}\right) / \sin\left(\frac{\sigma - \sigma_1}{2}\right)$, which becomes indeterminate for $\sigma = \sigma_1$, differentiating numerator and denominator with respect to σ , we have

$$\tan\left(\frac{\phi - \omega}{2}\right) = \sin \sigma_1 \left(\frac{\partial f}{\partial \sigma} + \frac{\partial f}{\partial c} \frac{\partial c}{\partial \sigma} \right)_{\sigma = \sigma_1},$$

or

$$\tan\left(\frac{\phi - \omega}{2}\right) = \sin \sigma_1 \left(\frac{\partial f}{\partial \sigma} + c' \frac{\partial f}{\partial c} \right)_{\sigma = \sigma_1},$$

where c' is an arbitrary constant. It is necessary to verify that this value of ϕ satisfies the equations (36), which is easily done*.

Transformation of Lie Another transformation of pseudospherical surfaces which, however, is analytical in character was introduced by Lie*. It is immediate when the surface is referred to asymptotic lines, or to any isothermal-conjugate system

the parameters in terms of which the surface is defined in an isothermal-conjugate, the parameters of the asymptotic system be given by

$$u + v = 2\alpha, \quad u - v = 2\beta$$

if of these curvilinear coordinates the linear elements of the surface and its spherical representation have the forms

$$ds^2 = a^2(d\alpha^2 + 2 \cos 2\omega d\alpha d\beta + d\beta^2),$$

$$d\sigma^2 = da^2 - 2 \cos 2\omega da d\beta + d\beta^2,$$

equation (23) takes the form

$$\frac{\partial^2 \omega}{\partial \alpha \partial \beta} = \sin \omega \cos \omega$$

In the form of this equation it is evident that if $\omega = \phi(\alpha, \beta)$ be a solution, so also is $\omega_1 = \phi(\alpha m, \beta/m)$, where m is any constant. From one pseudospherical surface we can obtain an infinity of others by the transformation of Lie. It should be remarked, however, that only the fundamental quantities of the new surfaces are given, and that the determination of the coordinates requires the solution of a Riccati equation which may be different from that for the given surface.

It is called attention to the fact that every Backlund transformation is a combination of transformations of Lie and Bianchi†

to prove this we effect the change of parameters (43) upon the surface (35) and obtain

$$\begin{cases} \frac{\partial}{\partial \alpha}(\theta + \omega) = \frac{1 + \cos \sigma}{\sin \sigma} \sin(\theta - \omega), \\ \frac{\partial}{\partial \beta}(\theta - \omega) = \frac{1 - \cos \sigma}{\sin \sigma} \sin(\theta + \omega) \end{cases}$$

* *Archiv for Mathematik og Naturvidenskab* Vol IV (1879) p 150

† Cf Bianchi Vol II, p 434, Darboux Vol III p 432

In particular, for a transformation of Bianchi we have

$$\frac{\partial}{\partial \alpha}(\theta + \omega) = \sin(\theta - \omega), \quad \frac{\partial}{\partial \beta}(\theta - \omega) = \sin(\theta + \omega)$$

Suppose that we have a pair of functions θ and ω satisfying these equations, and that we effect upon them the Lie transformation for which m has the value $(1 + \cos \sigma)/\sin \sigma$. This gives

$$\begin{aligned} \omega_1 &= \omega \left(\frac{1 + \cos \sigma}{\sin \sigma} \alpha, \frac{1 - \cos \sigma}{\sin \sigma} \beta \right), \\ \theta_1 &= \theta \left(\frac{1 + \cos \sigma}{\sin \sigma} \alpha, \frac{1 - \cos \sigma}{\sin \sigma} \beta \right) \end{aligned}$$

As these functions satisfy (44), they determine a transformation B_σ . But θ_1 may be obtained from ω_1 by effecting upon the latter an inverse Lie transformation, denoted by L_σ^{-1} , upon this result a Bianchi transformation, $B_{\pi/2}$, and then a direct Lie transformation, L_σ . Hence we may write symbolically

$$B_\sigma = L_\sigma^{-1} B_{\pi/2} L_\sigma,$$

which may be expressed thus

*A Backlund transformation B_σ is the transform of a Bianchi transformation by means of a Lie transformation L_σ .**

EXAMPLES

1 The asymptotic lines on a pseudospherical surface are curves of constant torsion

2 Every surface whose asymptotic lines are of the same length as their spherical images is a pseudospherical surface of curvature -1

3 Show that on the pseudosphere, defined by (9), the curves

$$b \cos \phi d\phi + \sin^2 \phi \sqrt{a^2 \sin^2 \phi - b^2} dv = 0,$$

where b is a constant, are geodesics, and find the radius of curvature of these curves

4 When the lineal element of a pseudospherical surface is in the parabolic form (iii) of (7), the surface defined by

$$x = x - a \frac{\partial x}{\partial u}, \quad y = y - a \frac{\partial y}{\partial u}, \quad z = z - a \frac{\partial z}{\partial u}$$

is pseudospherical (cf § 76), it is a Bianchi transform of the given surface

* Spherical surfaces admit of transformations similar to those of Lie and Backlund. The latter are imaginary but such combinations of them can be made that the resulting surface is real. For a complete discussion of these the reader is referred to chap. v of the *Lezioni* of Bianchi.

helicoids

$$x = u \cos v, \quad y = u \sin v, \quad z = \int \sqrt{\frac{1}{a - k^2 u^2} - \frac{k^2}{u^2}} - 1 \, du + kv,$$

, k are constants, are spherical surfaces

helicoid whose meridian curve is the tractrix is called the *surface of Dini*. Equations when $\sin \sigma$ denotes the helicoidal parameter and $\cos \sigma$ the cosine of the segment of the tangent between the curve and its axis. Show surface is pseudospherical

curves tangent to the joins of corresponding points on a pseudospherical surface on a Backlund transform are geodesics only when $\sigma = \pi/2$.

Let S be a pseudospherical surface and S_1 a Bianchi transform by means of θ (§ 119) Show that

$$\begin{aligned} X_1 &= \cos \omega (\cos \theta X_1 + \sin \theta X_2) - \sin \omega X, \\ X_2 &= \sin \omega (\cos \theta X_1 + \sin \theta X_2) + \cos \omega X, \\ X' &= \sin \theta X_1 - \cos \theta X_2, \end{aligned}$$

X_2, X are direction-cosines, with respect to the x -axis, of the tangents s of curvature on S and of the normal to S , and X'_1, X'_2, X' are the directions for S_1

W-surfaces Fundamental quantities Minimal surfaces, surfaces of constant curvature possess, in common with any other surfaces, the property that each of the principal curvatures is a function of the other. Surfaces of this kind were studied in detail by Weingarten,* and, in consequence, are called *Weingarten surfaces*, or simply *W-surfaces*. Since the principal curvatures of surfaces of revolution and of the general helicoids are functions of a single parameter (§§ 46, 62), these are *W* surfaces. We find other surfaces of this kind, but now we consider properties which are common to *W*-surfaces.

If a surface S is referred to its lines of curvature, the fundamental equations may be given the form

$$\frac{\partial \log \sqrt{E}}{\partial v} = \frac{1}{\rho - \rho_1} \frac{\partial \rho_1}{\partial v}, \quad \frac{\partial \log \sqrt{G}}{\partial u} = \frac{1}{\rho_1 - \rho_-} \frac{\partial \rho}{\partial u}$$

relation exists between ρ_1 and ρ , as

$$f(\rho_1, \rho_-) = 0,$$

Integration of equations (45) is reducible to quadratures, thus

$$\sqrt{E} = U e^{\int \frac{d\rho_1}{\rho - \rho_1}}, \quad \sqrt{G} = V e^{\int \frac{d\rho}{\rho_1 - \rho_-}},$$

* Crelle Vol LXII (1863) pp 160-173

where U and V are functions of u and v respectively. Without changing the parametric lines the parameters can be so chosen* that the above expressions reduce to

$$(47) \quad \sqrt{\mathcal{E}} = e^{\int \frac{d\rho_1}{\rho_2 - \rho_1}}, \quad \sqrt{\mathcal{G}} = e^{\int \frac{d\rho_2}{\rho_1 - \rho_2}}$$

Thus $\mathcal{E}$ and $\mathcal{G}$ are expressible as functions of ρ_1 or ρ_2 , and consequently they are functions of one another. This relation becomes more clear when we introduce an additional parameter κ defined by

$$(48) \quad \kappa = e^{\int \frac{d\rho_1}{\rho_1 - \rho_2}}$$

By the elimination of ρ_2 from this equation and (46) we have a relation of the form

$$\rho_1 = \phi(\kappa)$$

When this value is substituted in (48) we obtain

$$\rho_2 = \phi(\kappa) - \kappa\phi'(\kappa),$$

where the accent indicates differentiation with respect to κ . From (47) it follows that

$$\sqrt{\mathcal{E}} = \frac{1}{\kappa}, \quad \sqrt{\mathcal{G}} = \frac{1}{\phi'}$$

When these values are substituted in the Gauss equation for the sphere (V, 24), the latter becomes

$$\frac{\partial}{\partial u} \left(\frac{\kappa\phi''}{\phi'^2} \frac{\partial \kappa}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{\phi'}{\kappa^2} \frac{\partial \kappa}{\partial v} \right) - \frac{1}{\kappa\phi'} = 0$$

This equation places a restriction upon the forms of κ and $\phi(\kappa)$, but it is the only restriction, for the Codazzi equations (45) are satisfied. Hence we have the theorem of Weingarten *

When one has an orthogonal system on the unit sphere for which the linear element is reducible to the form

$$(49) \quad d\sigma^2 = \frac{du^2}{\kappa} + \frac{dv^2}{\phi'^2(\kappa)},$$

there exists a W-surface whose lines of curvature are represented by this system and whose principal radii are expressed by

$$(50) \quad \rho_1 = \phi(\kappa), \quad \rho_2 = \phi(\kappa) - \kappa\phi'(\kappa)$$

the coordinates of the sphere, namely X, Y, Z , are known in terms of u and v , the determination of the W -surface with this notation reduces to quadratures. For, from the formulas (IV, 32) we have

$$x = - \int \rho_1 \frac{\partial X}{\partial u} du + \rho_2 \frac{\partial X}{\partial v} dv,$$

$$y = - \int \rho_1 \frac{\partial Y}{\partial u} du + \rho_2 \frac{\partial Y}{\partial v} dv,$$

$$z = - \int \rho_1 \frac{\partial Z}{\partial u} du + \rho_2 \frac{\partial Z}{\partial v} dv$$

Right-hand members of these equations are exact differentials, the Codazzi equations (45) have been satisfied. If X, Y , are not known, their determination requires the solution of a differential equation. The relation between the radii of the form (46) obtained by eliminating κ from equations (50)

is found readily that the fundamental quantities for the surface have the values

$$\begin{cases} E = \frac{\phi^2}{\kappa^2}, & F = 0, & G = \left(\frac{\phi - \kappa \phi'}{\phi'} \right)^2, \\ D = \frac{\phi}{\kappa^2}, & D' = 0, & D'' = \frac{\phi - \kappa \phi'}{\phi'^2} \end{cases}$$

From (48), (50), and (51) we obtain

$$\sqrt{E} = \rho_1 e^{\int \frac{d\rho_1}{\rho - \rho_1}}, \quad \sqrt{G} = \rho e^{\int \frac{d\rho_2}{\rho_1 - \rho_2}}$$

Under the quadratic form

$$\frac{1}{H} [(F D' - F' D) du' + (F D'' - G D) dudv + (F D'' - G' D') dv'],$$

when equated to zero defines the lines of curvature. When the lines are parametric, this quadratic form is reducible by (IV, 74) to

$$\sqrt{G} \mathcal{E}(\rho_1 - \rho) dudv$$

consequence of (47) this is further reducible for H -surfaces to (53). Since the curvature of this latter form is zero, the curvature also is zero, and consequently (§ 135) the form (53) is reducible to quadratures. Hence we have the theorem of Lie: *lines of curvature of a H -surface can be found by quadratures*

124 Evolute of a W -surface The evolute of a W -surface possesses several properties which are characteristic. Referring to the results of § 75, we see that by means of (52) the linear elements of the sheets of the evolute of a W -surface are reducible to the form

$$(54) \quad \begin{cases} ds_1^2 = d\rho_1^2 + e^2 \int \frac{d\rho_1}{\rho_1 - \rho_2} dv^2, \\ ds_2^2 = d\rho_2^2 + e^2 \int \frac{d\rho_2}{\rho_2 - \rho_1} du^2, \end{cases}$$

or, in terms of κ ,

$$(55) \quad \begin{cases} ds_1^2 = \phi'^2 d\kappa^2 + \kappa^2 dv^2, \\ ds_2^2 = \kappa^2 \phi'^2 d\kappa^2 + \phi'^2 du^2 \end{cases}$$

From these results and the remarks of § 46 we obtain at once the following theorem of Weingarten

Each surface of center of a W surface is applicable to a surface of revolution whose meridian curve is determined by the relation between the radii of the given surface

We have also the converse theorem, likewise due to Weingarten

If a surface S_1 be applicable to a surface of revolution, the tangents to the geodesics on S_1 corresponding to the meridians of the surface of revolution are normal to a family of parallel W -surfaces, if S_1 be deformed in any manner whatever, the relation between the radii of these W -surfaces is unaltered

In proving this theorem we apply the results of § 76. If the linear element of S_1 be

$$ds_1^2 = du^2 + U^2 dv^2,$$

the principal radii of S are given by

$$(56) \quad \rho_1 = u, \quad \rho_2 = u - \frac{U}{U'}$$

Since both are functions of a single parameter, a relation exists between them which depends upon U alone, and consequently is unaltered in the deformation of S_1 .

From (V, 99) the projections upon the moving trihedral for S_1 of a displacement of a point on the complementary surface S_2 are

$$d\left(u - \frac{U}{U'}\right), \quad 0, \quad (q du + q_1 dv) \frac{U}{U'}$$

quence of formulas (V, 48, 75) the expression $U(q du + q_1 dv)$ act differential, which will be denoted by $d\omega$. Hence the element of S_2 is

$$ds_2^2 = U^2 \left(\frac{1}{U'} \right)^{1/2} du^2 + \frac{1}{U'^2} dv^2,$$

which it follows that S_2 also is applicable to a surface of

ast theorem of § 75 may be stated thus

necessary and sufficient condition that the asymptotic lines on
ices of center S_1, S_2 of a surface S correspond is that S be
face, in this case to every conjugate system on S_1 or S_2 there
nds a conjugate system on the other

(V, 98, 98') it follows that when S is a W -surface, and this case, we have

$$K_1 K_2 = \frac{1}{(\rho_1 - \rho_2)^2}$$

t corresponding points the curvature is of the same kind
ceptional form of equation (46) is afforded by the case where
both of the principal radii is constant. For the plane both
infinite, for a circular cylinder one is infinite and the other
finite constant value. The sphere is the only surface with both
finite and constant. For, if ρ_1 and ρ_2 are different constants,
) it follows that $\mathcal{E}$ and $\mathcal{F}$ are functions of u and v respec-
tively which is true only of developable surfaces. When one of the
infinite, the surface is developable. There remains the case
where has a finite constant value, then S is a canal surface (§ 29).
Considering the last case we take

$$\rho_2 = a,$$

on (48), we have

$$\rho_1 = \kappa + a,$$

linear element of the sphere is

$$d\sigma^2 = \frac{du^2}{\kappa} + dv^2$$

ely, when the linear element of the sphere is reducible to
a, the curves on the sphere represent the lines of curvature
infinity of parallel canal surfaces

125 Surfaces of constant mean curvature For surfaces of constant total curvature the relation (46) may be written

$$\rho_2 = \frac{c}{\rho_1},$$

where c denotes a constant. When this value is substituted in (48) we have, by integration,

$$(59) \quad \rho_1 = \sqrt{\kappa^2 + c}, \quad \rho_2 = \frac{c}{\sqrt{\kappa^2 + c}},$$

so that the linear element of the sphere is

$$(60) \quad d\sigma^2 = \frac{du^2 + (\kappa^2 + c)dv^2}{\kappa^2}$$

Conversely, when we have an orthogonal system on the sphere for which the linear element is reducible to the form (60), it serves for the representation of the lines of curvature of a surface of constant curvature, and of an infinity of parallel surfaces.

When c is positive, two of these parallel surfaces have constant mean curvature, as follows from the theorem of Bonnet (§ 73). In fact, the radii of these surfaces are

$$(61) \quad \rho_1 = \sqrt{\kappa^2 + c} \pm \sqrt{c}, \quad \rho_2 = \frac{c}{\sqrt{\kappa^2 + c}} \pm \sqrt{c}$$

If we put

$$(62) \quad c = a^2, \quad \kappa = a \operatorname{csch} \omega,$$

and replace u by au , the linear element (60) becomes

$$d\sigma^2 = \sinh^2 \omega du^2 + \cosh^2 \omega dv^2$$

In like manner, if we replace u by au , v by av , and take

$$(63) \quad c = a^2, \quad \kappa = a \operatorname{sech} \omega,$$

the linear element of the sphere is

$$d\sigma_1^2 = \cosh^2 \omega du^2 + \sinh^2 \omega dv^2$$

For the values (62) we have, from (61),

$$(64) \quad \rho_1 = \frac{ae^{\pm \omega}}{\sinh \omega}, \quad \rho_2 = \pm \frac{ae^{\pm \omega}}{\cosh \omega},$$

and the linear elements of the corresponding surfaces are

$$(65) \quad ds^2 = a^2 e^{\pm 2\omega} (du^2 + dv^2)$$

er, for the values (63) the radii have the values

$$\rho_1 = \pm \frac{ae^{\pm\omega}}{\cosh \omega}, \quad \rho_2 = \frac{ae^{\pm\omega}}{\sinh \omega},$$

linear elements are the same (65). In each case the mean curvature is $\pm 1/a$. We state these results in the following form: *Lines of curvature upon a surface of constant mean curvature in an isothermic system, the parameters of which can be chosen so that the linear element has one of the forms (65), where ω is a function of the equation*

$$\frac{\partial^2 \omega}{\partial u^2} + \frac{\partial^2 \omega}{\partial v^2} + \sinh \omega \cosh \omega = 0$$

ly, each solution of this equation determines two pairs of surfaces of constant mean curvature $\pm 1/a$, whose lines of curvature correspond, and for which the radii ρ_1, ρ_2 of one surface are reciprocal to the radii of ρ_2, ρ_1 of the applicable surface

It can be shown that if $\omega = \phi(u, v)$ is a solution of equation (67), then

$$\omega_1 = \phi(u \cos \sigma - v \sin \sigma, u \sin \sigma + v \cos \sigma),$$

where σ is any constant whatever. Hence there exists for spherical surfaces a transformation analogous to the Lie transformation of surfaces. This transformation can be given a geometric interpretation if it is considered in connection with the surfaces of constant mean curvature parallel to the spherical surfaces. Let S_1 denote the surface with the linear element

$$ds_1^2 = a^2 e^{2\omega_1} (du^2 + dv^2)$$

then

$$u_1 = u \cos \sigma - v \sin \sigma, \quad v_1 = u \sin \sigma + v \cos \sigma,$$

equation (68) becomes $\omega_1 = \phi(u_1, v_1)$, and (69) reduces to

$$ds_1^2 = a^2 e^{2\omega_1} (du_1^2 + dv_1^2)$$

If we make a point (u, v) on S with the linear element (65), then the positive sign is taken, correspond to the point (u_1, v_1) on the surfaces are applicable, and to the lines of curvature on S , $v = \text{const}$ on S correspond on S_1 the curves

$$u \cos \sigma - v \sin \sigma = \text{const}, \quad u \sin \sigma + v \cos \sigma = \text{const}$$

But the latter cut the lines of curvature $u = \text{const}$, $v = \text{const}$ on S_1 under the angle σ . Moreover, the corresponding principal radii of S and S_1 are equal at corresponding points. Hence we have the following theorem of Bonnet *

A surface of constant mean curvature admits an infinity of applicable surfaces of the same kind with preservation of the principal radii at corresponding points, and the lines of curvature on one surface correspond to lines on the other which cut the lines of curvature under constant angle

Weingarten has considered the W -surfaces whose lines of curvature are represented on the sphere by geodesic ellipses and hyperbolas. In this case the linear element of the sphere is reducible to the form (§ 90)

$$d\sigma^2 = \frac{du^2}{\sin^2 \frac{\omega}{2}} + \frac{dv^2}{\cos^2 \frac{\omega}{2}}$$

Comparing this with (49), we have

$$\kappa = \sin \frac{\omega}{2}, \quad \phi' = \cos \frac{\omega}{2},$$

from which it follows that

$$\phi = \frac{\omega + \sin \omega}{4}$$

Hence

$$(71) \quad \rho_1 = \frac{\omega + \sin \omega}{4}, \quad \rho_2 = \frac{\omega - \sin \omega}{4},$$

and the relation between the radii is found, by the elimination of ω , to be

$$(72) \quad 2(\rho_1 - \rho_2) = \sin 2(\rho_1 + \rho_2) \quad \dagger$$

* Memoire sur la theorie des surfaces applicables sur une surface donnee. *Journal de l'Ecole Polytechnique*, Cahier 42 (1867) pp 72 et seq. In this memoir Bonnet solves completely the problem of finding applicable surfaces with corresponding principal radii equal. When a surface possesses an infinity of applicable surfaces of this kind its lines of curvature form an isothermal system.

† Darboux (Vol. III, p. 373) proves that these surfaces may be generated as follows. Let C and C_1 be two curves of constant torsion differing only in sign. The locus of the mid points M of the join of any points P and P_1 of these curves is a surface of translation. If a line be drawn through M parallel to the intersection of the osculating planes of C and C_1 at P and P_1 , this line is normal to a W surface of the above type for all positions of M .

Ruled *W*-surfaces We conclude the present study of
 es with the solution of the problem

*termine the *W*-surfaces which are ruled*

problem was proposed and solved simultaneously by
 1* and Dini† We follow the method of the latter

106, 107 we found that when the linear element of a
 rface is in the form

$$ds^2 = du^2 + [(u - \alpha)^2 + \beta^2] dv^2,$$

essions for the total and mean curvatures are

$$K = -\frac{\beta^2}{g^4}, \quad K_m = -\frac{rg^2 + \beta'(u - \alpha) + \beta\alpha'}{g^3},$$

is a function of v at most, and

$$g^2 = (u - \alpha)^2 + \beta^2$$

der that a relation exist between the principal radii it is
 y and sufficient that the equation

$$\frac{\partial}{\partial u} K \frac{\partial}{\partial v} K_m - \frac{\partial}{\partial v} K \frac{\partial}{\partial u} K_m = 0$$

fied identically If the above values be substituted, the
 equation reduces to

$$\begin{aligned} & \frac{-\alpha}{\beta} \frac{\partial}{\partial v} \left[\frac{rg^2 + \beta'(u - \alpha) + \beta\alpha'}{g^3} \right] \\ & - \left\{ \frac{\beta^2 \beta' - (u - \alpha)[rg^2 + 2\beta'(u - \alpha) + 3\beta\alpha']}{g^5} \right\} \frac{\partial}{\partial v} \frac{g^2}{\beta} = 0 \end{aligned}$$

is an identical equation, it is true when $u = \alpha$, in which
 reduces to $\beta' = 0$ Hence β is a constant and the above
 becomes

$$r'(u - \alpha)^2 + r'\beta + \beta\alpha'' = 0$$

is equation must be true independently of the value of u
 and α'' are zero Therefore we have

$$\alpha = cv + d, \quad \beta = e, \quad r = k,$$

d, e, k are constants

linear element is

$$ds^2 = du^2 + [(u - cv - d)^2 + e^2] dv^2$$

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In order to interpret this result we calculate the expression for the tangent of the angle which the generators $v = \text{const.}$ make with the line of striction

$$u - cv - d = 0$$

From (III, 24) we have

$$\tan \theta = \frac{e}{c},$$

consequently the angle is constant. Conversely, if θ and the parameter of distribution β be constant, α has the form (73). Hence we have the theorem

A necessary and sufficient condition that a ruled surface be a W-surface is that the parameter of distribution be constant and that the generators be inclined at a constant angle to the line of striction, which consequently is a geodesic

EXAMPLES

- 1 Show that the helicoids are W surfaces
- 2 Find the form of equation (40), when the surface is minimal, and show that each conformal representation of the sphere upon the plane determines a minimal surface
- 3 Show that the tangents to the curves $v = \text{const.}$ on a spherical surface with the linear element (1) of (6) are normal to a W surface for which

$$\rho_2 - \rho_1 = \cot \frac{\rho_1}{a}$$

- 4 The helicoids are the only W surfaces which are such that the curves $\rho_1 = \text{const.}$ meet the lines of curvature under constant angle (cf Ex 23, p 188)
- 5 The asymptotic lines on the surfaces of center of a surface for which $\rho_1 + \rho_2 = \text{const.}$ correspond to the minimal lines on the spherical representation of the surface, and, when $\rho_1 - \rho_2 = \text{const.}$, to a rectangular system on the sphere

127 Spherical representation of surfaces with plane lines of curvature in both systems Surfaces whose lines of curvature in one or both systems are plane curves have been an object of study by many geometers. Since the tangents to a line of curvature and to its spherical representation at corresponding points are parallel, a plane line of curvature is represented on the sphere by a plane curve, that is, a circle, and conversely, a line of curvature is plane when its spherical representation is a circle

Consider first the determination of surfaces with plane curvature in both systems from the point of view of spherical representation * To this end we must find orthogonal systems of circles on the sphere If two circles cut one another orthogonally, the plane of each must pass through the center of the other Hence the planes of the circles of one system pass through a point in the plane of each circle of the second system, and consequently the planes of each family form a pencil, the two axes being polar reciprocal with respect to the sphere †

Consider separately the two cases I, when one axis is tangent to the sphere, and therefore the other is tangent at the same point and perpendicular to it, II, when neither is tangent

I We take the center of the unit sphere for origin O , the x -axes parallel to the axes of the pencils, and let the coordinates of the point of contact be $(0, 0, 1)$ The equations of the planes of the pencils may be put in the form

$$x + u(z - 1) = 0, \quad y + v(z - 1) = 0,$$

where u and v are the parameters of the respective families of planes. These equations be solved simultaneously with the equation of the sphere, and, as usual, X, Y, Z denote coordinates of the point, we have

$$X = \frac{2u}{u^2 + v + 1}, \quad Y = \frac{2v}{u^2 + v^2 + 1}, \quad Z = \frac{u^2 + v^2 - 1}{u^2 + v + 1}$$

The linear element of the sphere is

$$d\sigma^2 = \frac{4(du^2 + dv^2)}{(u^2 + v + 1)^2}$$

II As in the preceding case, we take for the z axis the axis perpendicular to the axes of the pencils, and for the x -axes we take lines through O parallel to the axes of the pencils

The coordinates of the points of meeting of the latter with the z axis are of the form $(0, 0, a), (0, 0, 1/a)$ The equations of the two pencils of planes could be written in forms

similar to (74), but the expressions for X , Y , Z will be found to be of a more suitable form if the equations of the families of planes be written

$$x - \frac{\tan u}{\sqrt{1-a^2}}(z-a) = 0, \quad y - \frac{a \tanh v}{\sqrt{1-a^2}}\left(z - \frac{1}{a}\right) = 0$$

Proceeding as in Case I, we find

$$(77) \quad \begin{cases} X = \frac{\sqrt{1-a^2} \sin u}{\cosh v + a \cos u}, \\ Y = -\frac{\sqrt{1-a^2} \sinh v}{\cosh v + a \cos u}, \\ Z = \frac{\cos u + a \cosh v}{\cosh v + a \cos u}, \end{cases}$$

and the linear element is

$$(78) \quad d\sigma^2 = \frac{(1-a^2)(du^2 + dv^2)}{(\cosh v + a \cos u)^2}$$

From the preceding discussion we have tacitly excluded the system of meridians and parallels. As before, the planes of the two families of circles form pencils, but now the axis of one pencil passes through the center of the sphere and the other is at infinity. Hence this case corresponds to the value zero for a in Case II. In fact, if we put $a = 0$ in (77), the resulting equations define a sphere referred to a system of meridians and parallels, namely

$$(79) \quad X = \frac{\sin u}{\cosh v}, \quad Y = -\frac{\sinh v}{\cosh v}, \quad Z = \frac{\cos u}{\cosh v}$$

Since the planes of the lines of curvature on a surface are parallel to the planes of their spherical images, the curves $v = \text{const}$ on a surface with the representation (79) lie in parallel planes, and the planes of the curves $u = \text{const}$ envelop a cylinder. These surfaces are called the *molding surfaces**. We shall consider them later.

128 Surfaces with plane lines of curvature in both systems
By a suitable choice of coordinate axes and parameters the expressions for the direction cosines of the normal to a surface with plane lines of curvature in both systems can be given one

* These surfaces were first studied by Monge, *Application de L'Analyse a la Géométrie*, § 17. Paris, 1849.

orms (75) or (77) For the complete determination of all of this kind it remains then for us to find the expression for the other tangential coordinate W , that is, the distance from the origin to the tangent plane. The linear element of arc in both cases is of the form

$$d\sigma^2 = \frac{du^2 + dv^2}{\lambda^2},$$

is such that

$$\frac{\partial^2 \lambda}{\partial u \partial v} = 0$$

In § 39) we see that the equation satisfied by W is

$$\frac{\partial^2 \theta}{\partial u \partial v} + \frac{\partial \log \lambda}{\partial v} \frac{\partial \theta}{\partial u} + \frac{\partial \log \lambda}{\partial u} \frac{\partial \theta}{\partial v} = 0$$

sequence of (80), if we change the unknown function in place with $\theta_1 = \lambda \theta$, the equation in θ_1 is of the form (80). The most general value for W is

$$W = \frac{U + V}{\lambda},$$

U and V are arbitrary functions of u and v respectively. We may take any surface with plane lines of curvature in both systems as the envelope of a family of planes whose equation is of the form

$$2ux + 2vy + (u^2 + v^2 - 1)z = 2(U + V),$$

$$\begin{aligned} & \sqrt{1-a} \sin ux - \sqrt{1-a^2} \sinh vy + (\cos u + a \cosh v)z \\ & = (U + V) \sqrt{1-a} \end{aligned}$$

expressions for the Cartesian coordinates of these surfaces can be found without quadrature by the methods of § 67. Thus, for the surface envelope of (81) we have to solve for x, y, z equations (81) and its derivatives with respect to u and v . The latter are

$$x + uz = U', \quad y + vz = V',$$

The accents indicate differentiation. We shall not carry out the integration, but remark that as each of these equations contains the parameter z they define the planes of the lines of curvature

From the form of (83) it is seen that these planes in each system envelop a cylinder, and that the axes of these two cylinders are perpendicular. This fact was remarked by Darboux, who also observed that equation (81) defines the radical plane of the two spheres

$$(84) \quad \begin{cases} x^2 + y^2 + z^2 + 2ux + (u^2 - \frac{1}{2})z = 2U, \\ x^2 + y^2 + z^2 - 2vy - (v^2 - \frac{1}{2})z = -2V \end{cases}$$

These are the equations of two one-parameter families of spheres, whose centers lie on the focal parabolas

$$(85) \quad \begin{cases} x_1 = -u, & y_1 = 0, & z_1 = \frac{1}{2}(\frac{1}{2} - u^2), \\ x_2 = 0, & y_2 = v, & z_2 = \frac{1}{2}(v^2 - \frac{1}{2}), \end{cases}$$

and whose radii are determined by the arbitrary functions U and V . The characteristics of each family are defined by its equation and the corresponding equation of the pair (83). Consequently the original surface is the locus of the point of intersection of the planes of these characteristics and the radical planes of the spheres.

Similar results follow for the equation (82), which defines the radical planes of two families of spheres whose centers are on the focal ellipse and hyperbola

$$(86) \quad \begin{cases} x_1 = \frac{1}{2}\sqrt{1-a^2}\sin u, & y_1 = 0, & z_1 = \frac{1}{2}\cos u, \\ x_2 = 0, & y_2 = \frac{1}{2}\sqrt{1-a^2}\sinh v, & z_2 = -\frac{1}{2}a\cosh v \end{cases}$$

When in particular $a = 0$, these curves of center are a circle and its axis.

From the foregoing results it follows that these surfaces may be generated by the following geometrical method due to Darboux *

Every surface with plane lines of curvature in two systems can be obtained from two singly infinite families of spheres whose centers lie on focal conics and whose radii vary according to an arbitrary law. The surface is the envelope of the radical plane of two spheres S and Σ , belonging to two different families. If one associate with S and Σ two infinitely near spheres S' and Σ' , the radical center of these four spheres describes the surface, and the radical planes of S and S' and of Σ and Σ' are the planes of the lines of curvature.

Surfaces with plane lines of curvature in one system.
of Monge When the lines of curvature in one system are, the curves on the sphere are a family of circles and orthogonal trajectories, and conversely Every system of d may be obtained from a system of circles and their orthogonal trajectories in a plane by a stereographic projection. Determination of such a system in the plane reduces to the solution of a Riccati equation (Ex 11, p 50) Since the circles are of constant geodesic curvature we have, in consequence of the first theorem of § 84, the theorem

Determination of all the surfaces with plane lines of curvature in one system requires the solution of a Riccati equation and quadratures

We shall discuss at length several kinds of surfaces with plane lines of curvature in one system, and begin with the case where the lines of curvature are geodesics They are consequently normal sections of the surface Their planes envelop a developable surface, called *envelope-developable*, and the lines of curvature in the other system are the orthogonal trajectories of these planes Conversely, if there is of any simple infinity of the orthogonal trajectories of a system of planes is a surface of the kind sought The planes cut the surface orthogonally, and consequently the lines of curvature and geodesics (§ 59) Since these are the osculating planes of the edge of regression of the developable, the orthogonal trajectories can be found by quadratures (§ 17)

Suppose that we have such a surface, and that C denotes one of the lines of curvature of the family of plane lines of curvature Let the coordinates of C be expressed in terms of the arc of the surface from a point of it, which will be denoted by v As the line of each plane line of curvature Γ is normal to C at its point of intersection with the latter, the coordinates of a point P of Γ with respect to the moving trihedral of C are $0, \eta, \zeta$ Since P describes an orthogonal trajectory of the planes, we must have (I, 82)

$$\frac{d\eta}{dv} + \frac{\zeta}{\tau} = 0, \quad \frac{d\zeta}{dv} - \frac{\eta}{\tau} = 0,$$

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where τ denotes the radius of torsion of C . If we change the parameter of C in accordance with the equation

$$(87) \quad v_1 = \int_0^v \frac{dv}{\tau},$$

the above equations become

$$\frac{d\eta}{dv_1} + \zeta = 0, \quad \frac{d\zeta}{dv_1} - \eta = 0$$

The general integral of these equations is

$$(88) \quad \eta = U_1 \cos v_1 - U_2 \sin v_1, \quad \zeta = U_1 \sin v_1 + U_2 \cos v_1,$$

where U_1 and U_2 are functions of the parameter u of points of Γ . When $v=0$ we have $v_1=0$, and so the curve Γ in the plane through the point $v=0$ of C has the equations $\eta = U_1$, $\zeta = U_2$. Hence the character of the functions U_1 and U_2 is determined by the form of the curve, and conversely, the functions U_1 and U_2 determine the character of the curve.

By definition (87) the function v_1 measures the angle swept out in the plane normal to C by the binormal of the latter, as this plane moves from $v=0$ to any other point. Hence equations (88) define the same curve, in this moving plane, for each value of v_1 , but it is defined with respect to axes which have rotated through the angle v_1 . Hence we have the theorem

Any surface whose lines of curvature in one system are geodesics can be generated by a plane curve whose plane rolls, without slipping, over a developable surface.

These surfaces are called the *surfaces of Monge*, by whom they were first studied. He proposed the problem of finding a surface with one sheet of the evolute a developable. It is evident that the above surfaces satisfy this condition. Moreover, they furnish the only solution. For, the tangents to a developable along an element lie in the plane tangent along this element, and if these tangents are normals to a surface, the latter is cut normally by this plane, and consequently the curve of intersection is a line of curvature. In particular, a molding surface (§ 127) is a surface of Monge with a cylindrical director-developable.

Since every curve in the moving plane of the lines of curvature generates a surface of Monge, a straight line in this plane

is a developable surface of Monge. For, all the normals to the surface along a generator lie in a plane (§ 25). Hence

necessary and sufficient condition that a curve Γ in a plane generate a surface of Monge as it moves, remaining normal to the curve, is that the line from a point of Γ to Q generate a developable

Molding surfaces. When the orthogonal trajectory C is a circle, the planes of the curves Γ are perpendicular to the radius of C , and consequently the director-developable is a cylinder. The right section is the plane evolute of C . The surface is a *surface* (§ 127), and all the lines of curvature of the section are plane curves, — involutes of the right section of the cylinder. Hence a molding surface may be generated by a cylinder whose plane rolls without slipping over a cylinder. We shall now apply the preceding formulas to this particular case.

Since $1/\tau$ is equal to zero, it follows from (88) that η and ζ are functions of u alone. If u be taken as a measure of the arc of the circle, we have, in all generality,

$$\eta = U, \quad \zeta = \int \sqrt{1 - U'^2} du,$$

where the function U determines the form of Γ . If we take the origin on the curve C for $z = 0$, and x_0, y_0 denote the coordinates of a point of C , the equations of the surface may be written

$$x = x_0 + U \cos v, \quad y = y_0 + U \sin v, \quad z = \int \sqrt{1 - U'^2} du,$$

where v denotes the angle which the principal normal to C makes with the z -axis.

Since $\frac{dx_0}{ds_0} = \sin v$ and $\frac{dy_0}{ds_0} = -\cos v$,

if ρ denote the radius of curvature of C , then $ds_0 = \rho dv$, and the equations of the surface can be put in the following form, given by

$$\begin{cases} x = U \cos v + \int \rho \sin v dv, \\ y = U \sin v - \int \rho \cos v dv, \\ z = \int \sqrt{1 - U'^2} du \end{cases}$$

The equations of the right section of the cylinder are

$$\bar{x} = x_0 + V \cos v = \int V' \cos v \, dv,$$

$$\bar{y} = y_0 + V \sin v = \int V' \sin v \, dv$$

In passing, we remark that surfaces of revolution are molding surfaces, whose director-cylinder is a line, this corresponds to the case $V = 0$

EXAMPLES

1 When the spherical representation of the lines of curvature of a surface isothermal and the curves in one family on the sphere are circles, the curves in the other family also are circles

2 If the lines of curvature in one system on a minimal surface are plane, then in the other system also are plane

3 Show that the surface

$$x = au + \sin u \cosh v, \quad y = v + a \cos u \sinh v, \quad z = \sqrt{1 - a^2} \cos u \cosh v,$$

is minimal and that its lines of curvature are plane. Find the spherical representation of these curves and determine the form of the curves

4 Show that the surface of Ex. 3 and the Enneper surface (Ex. 18, p. 209) are the only minimal surfaces with plane lines of curvature

5 When the lines of curvature in one system lie in parallel planes, the surface is of the molding type

6 A necessary and sufficient condition that the lines of curvature in one system on a surface be represented on the unit sphere by great circles is that it be a surface of Monge

7 Derive the expressions for the point coordinates of a molding surface by method of § 67

131 Surfaces of Joachimsthal Another interesting class of surfaces with plane lines of curvature in one system are those for which all the planes pass through a straight line. Let one of the lines of curvature be denoted by Γ , and one of the other systems by C . The developable enveloping the surface along the latter for its elements the tangents to the curves Γ at their point of intersection with C . Since these elements lie in the planes of the curves Γ , the developable is a cone with its vertex on the line through which all these planes pass. This cone is tangent to the surface along C , and its elements are orthogonal to the latter. Consequently C is the intersection of the surface and a sphere.

t the vertex of the cone which cuts the surface orthogonally we have the following result, due to Joachimsthal *

the lines of curvature in one system lie in planes passing a line D , the lines of curvature in the second system lie on whose centers are on D and which cut the surface orthogonally.

surfaces are called *surfaces of Joachimsthal*. Each of the first system is an orthogonal trajectory of the circles of the spheres are cut by its plane. Therefore, in order to find the equations of such a surface, we consider first the orthogonal trajectories of a family of circles whose centers are on a line. Let η be taken for the η -axis, the circles are defined by

$$\xi = r \sin \theta, \quad \eta = r \cos \theta + u,$$

where r denotes the radius, θ the angle which the latter makes with the η -axis, and u the distance of the center from the origin. r is a function of u , and θ is independent of u . In order that the equations may define an orthogonal trajectory of the family, θ must be such a function of u that

$$\cos \theta \frac{\partial \xi}{\partial u} - \sin \theta \frac{\partial \eta}{\partial u} = 0,$$

$$r \frac{d\theta}{du} - \sin \theta = 0$$

Integration we have

$$\tan \frac{\theta}{2} = r e^{\int \frac{du}{r}},$$

where r denotes the constant of integration

each section of a surface of Joachimsthal by a plane through its axis is an orthogonal trajectory of a family of circles whose centers are on this axis, the equations of the most general of this kind are of the form

$$x = r \sin \theta \cos v, \quad y = r \sin \theta \sin v, \quad z = u + r \cos \theta,$$

where v denotes the angle which the plane through a point and the axis makes with the plane $y = 0$, and θ is given by (90), in which now V is a function of v

When V is constant θ is a function of u alone, and the surface is one of revolution. For other forms of V the geometrical generation of the surfaces is given by the theorem

Given the orthogonal trajectories of a family of circles whose centers lie on a right line D , if they be rotated about D through different angles, according to a given law, the locus of the curves is surface of Joachimsthal

132 Surfaces with circular lines of curvature We consider next surfaces whose lines of curvature in one system are circles. Let σ denote the constant angle between the plane of the circle C and the tangent planes to the surface along C (cf § 59), ρ the radius of normal curvature in the direction of C , and r the radius of the latter. Now equation (IV, 17) may be written

$$(91) \quad r = \rho \sin \sigma$$

As an immediate consequence we have the theorem

A necessary and sufficient condition that a plane line of curvature be a circle is that the normal curvature of the surface in its direction be the same at all of its points

Since the normals to the surface along C are inclined to its plane under constant angle, they form a right circular cone whose vertex is on the axis of C . Moreover, the cone cuts the surface at right angles, and consequently the sphere of radius ρ and center at the vertex of the cone is tangent to the surface along C . Hence the surface is the envelope of a family of spheres of variable or constant radius, whose centers lie on a curve.

Conversely, we have seen in § 29 that the characteristics of the family of spheres

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = h^2,$$

where x, y, z are the coordinates of a curve expressed in terms of arc, and h is a function of the same parameter, are circles of radii

$$(92) \quad r = h(1 - h'^2)^{\frac{1}{2}},$$

whose axes are tangent to the curve of centers and whose centers have the coordinates

$$(93) \quad x_1 = x - \alpha h h', \quad y_1 = y - \beta h h', \quad z_1 = z - \gamma h h',$$

β, γ are the direction-cosines of the axis, and the accent is differentiation. The normals to the envelope along a principal form a cone, and consequently these circles are lines of curvature upon it. Hence

A necessary and sufficient condition that the lines of curvature on a surface be circles is that the surface be the envelope of a single family of spheres, the locus of whose centers is a curve, the radius determined by an arbitrary law

From equations (91), (92) it follows that $R' = \cos \sigma$. Hence the principal are geodesics only when R is constant, that is, for canal surfaces (§ 29). In this case, as is seen from (92), all the circles have

equal radius a when

$$R^2 = (s + c)^2 + a^2,$$

where s is the arc of the curve of centers and c is a constant of integration. Now equations (93) become

$$x = (s + c)\alpha, \quad y = (s + c)\beta, \quad z = (s + c)\gamma,$$

which are the equations also of an involute of the curve of centers. This result may be stated thus*

A surface may be generated by a string being unwound from a curve in such a way that its moving end M generates an involute of the curve, and if at M a circle is described whose center is M and whose plane is normal to the curve, then as the string is unwound this circle generates a surface composed of equal circles for lines of curvature.

The locus of the centers of the spheres enveloped by a surface is a curve on one sheet of the evolute of the surface, and the radius of a sphere is the radius of normal curvature in the direction of the principal circle. Consequently this radius is a function of the arc of the spheres. Conversely, from § 75, we have that the locus is a curve $H_2 = 0$, and consequently

$$(\rho_1 - \rho) \frac{\partial \rho}{\partial v} = 0$$

* Cf. Bianchi, Vol. II, p. 272

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Excluding the case of the sphere, we have that ρ_2 is a function of u alone. From the formulas of Rodrigues (IV, 32),

$$\frac{\partial x}{\partial v} = -\rho_2 \frac{\partial X}{\partial v}, \quad \frac{\partial y}{\partial v} = -\rho_2 \frac{\partial Y}{\partial v}, \quad \frac{\partial z}{\partial v} = -\rho_2 \frac{\partial Z}{\partial v},$$

we have, by integration,

$$x = -\rho_2 X + U_1, \quad y = -\rho_2 Y + U_2, \quad z = -\rho_2 Z + U_3$$

Hence the points of the surface lie on the spheres

$$(x - U_1)^2 + (y - U_2)^2 + (z - U_3)^2 = \rho_2^2,$$

and the spheres are tangent to the surface

Since the normals to a surface along a circular line of curvature form a cone of revolution, the second sheet of the evolute is the envelope of a family of such cones. The characteristics of such family are conics. Hence we have the theorem

A necessary and sufficient condition that one sheet of the evolute of a surface be a curve is that the surface be the envelope of a single infinite family of spheres, the second focal sheet is the locus of a family of conics

133 Cyclides of Dupin From the preceding theorem it results that if also the second sheet of the evolute of a surface be a curve it is a conic, and then the first sheet also is a conic. Moreover, these conics are so placed that the cone formed by joining any point of one conic to all the points of the other is a cone of revolution. A pair of focal conics is characterized by this property. And so we have the theorem

*A necessary and sufficient condition that the lines of curvature of both families be circles is that the sheets of the evolute be a pair of focal conics **

These surfaces are called the *cyclides of Dupin*. They are the envelopes of two one-parameter families of spheres, and all such envelopes are cyclides of Dupin. A sphere of one family touches each sphere of the other family. Consequently the spheres of which the cyclide is the envelope are tangent to three spheres.

We shall prove the converse theorem of Dupin †

The envelope of a family of spheres tangent to three fixed spheres is a cyclide

* Cf. Ex. 19, p. 188

† *Applications de géométrie et de mécanique* pp. 200-210. Paris, 1822

plane determined by the centers of the three spheres cuts L in three circles. If any point on the circumference C , normal to these circles, be taken for the pole of a transformation-reciprocal radii (cf. § 80), C is transformed into a straight line. Since angles are preserved in this transformation, the three spheres are changed into three spheres whose centers are on L . The envelope of a family of spheres tangent to these three spheres is a torus with L as axis. Hence the given envelope is transformed into a torus. However, the latter surface is the envelope of a family of spheres whose centers lie on L . Therefore, if the transformation be reversed, we have a second family of spheres tangent to the envelope, and so the latter is a cyclide of Dupin. We shall now find the equations of these surfaces.

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) denote the coordinates of the points on the focal conics which are the curves of centers of the spheres, R_1, R_2 the radii of the spheres. The condition of tangency is

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = (R_1 + R_2)^2$$

Consider first the case where the evolute curves are the focal conics defined by (85). Now equation (94) reduces to

$$\frac{1}{4}(u + v^2 + 1)^2 = (R_1 + R_2)^2$$

Since R_1 and R_2 are functions of u and v respectively, this equation is equivalent to

$$R_1 = \frac{1}{2}\left(u^2 + \frac{1}{2} + a\right), \quad R_2 = \frac{1}{2}\left(v^2 + \frac{1}{2} - a\right),$$

where a is an arbitrary constant whose variation gives parallel

surfaces. By the method of § 132 we find that the coordinates (ξ, η, ζ) of the centers of the circular lines of curvature $u = \text{const}$ and the $v = \text{const}$ are

$$\xi = \frac{u}{2(1+u)} \left[u^2 + \frac{1}{2} + a - 2(1+u) \right],$$

$$\eta = 0,$$

$$\zeta = \frac{1}{2(1+u)} \left[u \left(u^2 + \frac{1}{2} + a \right) + \left(u^2 - \frac{1}{2} \right) (1+u) \right],$$

$$\rho = \frac{1}{2} \left(u^2 + \frac{1}{2} + a \right) (1+u^2)^{-\frac{1}{2}}$$

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Hence if P be a point on the circle and θ denote the angle which the radius to P makes with the positive direction of the normal to the parabola (85), the coordinates of P are

$$\bar{x} = \xi + \frac{\rho u}{\sqrt{1+u^2}} \cos \theta, \quad \bar{y} = \rho \sin \theta, \quad \bar{z} = \zeta - \frac{\rho}{\sqrt{1+u^2}} \cos \theta$$

This surface is algebraic and of the third order

If the evolute curves are the focal ellipse and hyperbola (86), we have

$$(96) \quad R_1 = \frac{1}{2}(a \cos u + \kappa), \quad R_2 = \frac{1}{2}(\cosh v - \kappa),$$

where κ is an arbitrary constant whose variation gives parallel surfaces. This cyclide of Dupin is of the fourth degree. When in particular the constant a is zero, the surface is the ordinary torus, or anchor ring.

134 Surfaces with spherical lines of curvature in one system. Surfaces with circular lines of curvature in one system belong evidently to the general class of surfaces with spherical lines of curvature in one system. We consider now surfaces of the latter kind.

Let S be such a surface referred to its lines of curvature, and in particular let the lines $v = \text{const}$ be spherical. The coordinates of the centers of the spheres as well as their radii are functions of v alone. They will be denoted by (V_1, V_2, V_3) and R . By Joachimsthal's theorem (§ 59) each sphere cuts the surface under the same angle at all its points. Hence for the family of spheres the expression for the angle is a function of v alone, we call it h .

Since the direction-cosines of the tangent to a curve $u = \text{const}$ are

$$\frac{1}{\sqrt{G}} \frac{\partial X}{\partial v}, \quad \frac{1}{\sqrt{G}} \frac{\partial Y}{\partial v}, \quad \frac{1}{\sqrt{G}} \frac{\partial Z}{\partial v},$$

when the linear element of the spherical representation is written $d\sigma^2 = \mathcal{E} du^2 + \mathcal{G} dv^2$, the coordinates of S are of the form

$$(97) \quad \begin{cases} x = V_1 + \frac{R \sin V}{\sqrt{G}} \frac{\partial X}{\partial v} + h \cos V, \\ y = V_2 + \frac{R \sin V}{\sqrt{G}} \frac{\partial Y}{\partial v} + h \cos V, \\ z = V_3 + \frac{R \sin V}{\sqrt{G}} \frac{\partial Z}{\partial v} + h \cos V \end{cases}$$

* For other geometrical constructions of the cyclides of Dupin the reader is referred to the article in the *Encyclopädie der Math. Wissenschaften* Vol. III 3 p. 290.

hesis X, Y, Z are the direction-cosines of the normal to S , ntly we must have

$$\sum X \frac{\partial x}{\partial u} = 0, \quad \sum X \frac{\partial x}{\partial v} = 0$$

lues of the derivatives obtained from (97) be reduced by (V, 22), and the results substituted in the above equation, first vanishes identically and the second reduces to

$$XV'_1 + YV'_2 + ZV'_3 + (R \cos V)' - R \sin V \sqrt{g} = 0,$$

e primes indicate differentiation with respect to v . When this condition is satisfied, equations (97) define a n which the curves $v = \text{const}$ are spherical. Hence

necessary and sufficient condition that the curves $v = \text{const}$ of an l system on the unit sphere represent spherical lines of curvature on a surface is that five functions of v , namely V_1, V_2, V_3 , be found which satisfy the corresponding equation (98)

ote that V_1, V_2, V_3 , and $R \cos V$ are determined by (98) within additive constants. A change of these constants rst three gives a translation of the surface. If $R \cos V$ be by a constant, we have a new surface parallel to the e. Hence *

lines of curvature in one system upon a surface be spherical, is true of the corresponding system on each parallel surface

equation (98) is homogeneous in the quantities $V'_1, V'_2, V'_3, V', R \sin V$, the latter are determined only to within a factor ay be a function of v . This function may be chosen so he spheres pass through a point. From these results we theorem of Dobiner †

each surface with spherical lines of curvature in one system associated an infinity of nonparallel surfaces of the same the same spherical representation of these lines of curvature hese surfaces there is at least one for which all the spheres ough a point. At corresponding points of the loci of the centres of two surfaces of the family the tangents are parallel

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If the values of x, y, z from (97) be substituted in the formula of Rodrigues (IV, 32),

$$(99) \quad \frac{\partial x}{\partial u} = -\rho_1 \frac{\partial X}{\partial u}, \quad \frac{\partial x}{\partial v} = -\rho_2 \frac{\partial X}{\partial v},$$

and similarly for y and z , we obtain by means of (V, 22),

$$\begin{aligned} -\rho_1 &= R \cos V + \frac{R \sin V}{\sqrt{\mathcal{E}\mathcal{G}}} \frac{\partial \sqrt{\mathcal{E}}}{\partial v}, \\ -\rho_2 &= R \cos V + \frac{(R \sin V)'}{\sqrt{\mathcal{G}}} + \frac{1}{\mathcal{G}} \sum V_1' \frac{\partial X}{\partial v} \end{aligned}$$

Conversely, when for a surface referred to its lines of curvature the principal radius ρ_1 is of the form

$$(100) \quad -\rho_1 = \phi_1(v) + \frac{\phi_2(v)}{\sqrt{\mathcal{E}\mathcal{G}}} \frac{\partial \sqrt{\mathcal{E}}}{\partial v},$$

where ϕ_1 and ϕ_2 are any functions whatever of v , the curve $v = \text{const}$ are spherical. For, by (V, 22),

$$\frac{\partial}{\partial u} \left(\frac{1}{\sqrt{\mathcal{G}}} \frac{\partial X}{\partial v} \right) = \frac{1}{\sqrt{\mathcal{E}\mathcal{G}}} \frac{\partial \sqrt{\mathcal{E}}}{\partial v} \frac{\partial X}{\partial u}$$

Consequently, from the first of (99), in which ρ_1 is given the above value, we obtain by integration

$$x = \phi_1 X + \phi \frac{1}{\sqrt{\mathcal{G}}} \frac{\partial X}{\partial v} + V_1,$$

where V_1 is a function of v alone. Similar results follow for y and z . As these expressions are of the form (97), we have the theorem

A necessary and sufficient condition that the lines of curvature $v = \text{const}$ be spherical is that ρ_1 be of the form (100)

EXAMPLES

1 If the lines of curvature in one system are plane and one is a circle, all are circles

2 When the lines of curvature in one family on a surface are circles, their spherical images are circles whose spherical centers constitute the spherical indicatrix of the tangents to the curve of centers of the spheres which are envelope by the given surface. Show also that each one parameter system of circles on the unit sphere represents the circular lines of curvature on an infinity of surfaces for one of which the circles are equal

the lines of curvature of a surface are parametric, and the curves $v = \text{const.}$, we have

$$\frac{1}{\rho_{\text{ge}}} = \frac{1}{R \sin V} + \frac{\cot V}{\rho_1},$$

ρ_1 , R denote the radii of geodesic curvature and normal curvature in the $v = \text{const.}$ and of the sphere respectively, and V denotes the angle under which the sphere cuts the surface

When a line of curvature is spherical, the developable circumscribing the surface along this line of curvature also circumscribes a sphere, and conversely, when a developable circumscribes a sphere, the line of curvature lies on a sphere with the latter (cf Ex 7, p 149)

Let S be a pseudospherical surface with the spherical representation (35) of its curvature. Show that a necessary and sufficient condition that the curves $v = \text{const.}$ be plane is

$$\frac{\partial}{\partial u} \left(\frac{1}{\sin \omega} \frac{\partial \omega}{\partial v} \right) = 0,$$

that in this case ω is given by

$$\cos \omega = \frac{V' - U'}{U^2 - V^2 - 1}$$

where U and V are functions of u and v respectively, which satisfy the conditions

$$U'^2 = U^4 + (a - 2)U^2 + b, \quad V'^2 = V^4 + aV^2 + (a + b - 1),$$

where a , b are constants, and the accent indicating differentiation, unless U' or V'

When the lines of curvature $v = \text{const.}$ upon a pseudospherical surface are plane, the linear element is reducible to the form

$$ds^2 = \frac{a^2 \tanh^2(u + v) du^2}{C - A \cosh 2u + B \sinh 2u} + \frac{a^2 \text{sech}^2(u + v) dv^2}{C + A \cosh 2v + B \sinh 2v - 1},$$

where A , B , C are constants. Find the expressions for the principal radii of curvature.

When the lines of curvature $v = \text{const.}$ on a spherical surface are plane, the linear element is reducible to

$$ds^2 = \frac{a^2 \cot^2(u + v) du^2}{A \sin 2u + B - 1} + \frac{a^2 \csc^2(u + v) dv^2}{A \sin 2v - B},$$

where A , B are constants. The surfaces of Exs 5 and 6 are called *the surfaces of constant curvature*.

GENERAL EXAMPLES

Find the lines of curvature and the asymptotic lines on a surface of constant curvature. Can be found by quadratures.

When the linear element of a pseudospherical surface is in the form (III) of (7), the transformations $x = cv$, $y = ae^{-u}$ determine a conformal representation of the surface on the xy -plane, which is such that any geodesic on the surface is represented on the xy -plane by a circle with its center on the x -axis, or by a line perpendicular to the x -axis.

3 When the linear elements of a developable surface, a spherical surface, and a pseudospherical surface are in the respective forms

$$ds^2 = du^2 + u^2 dv^2, \quad ds^2 = a^2 (du^2 + \sin^2 u dv^2), \quad ds^2 = a^2 (du^2 + \sinh^2 u dv^2),$$

the finite equations of the geodesics are respectively

$$Au \cos v + Bu \sin v + C = 0, \quad A \tan u \cos v + B \tan u \sin v + C = 0, \\ A \tanh u \cos v + B \tanh u \sin v + C = 0,$$

where A, B, C are constants, if the coefficients of A and B are in any case equated to x and y , the resulting equations define a correspondence between the surface and the plane such that geodesics on the former correspond to straight lines on the latter. Find the expression for each linear element in terms of x and y as parameters.

4 Each surface of center of a pseudospherical surface is applicable to the catenoid.

5 The asymptotic lines on the surfaces of center of a surface of constant mean curvature correspond to the minimal lines on the latter.

6 Surfaces of constant mean curvature are characterized by the property that if $u = \text{const}$, $v = \text{const}$ are the minimal curves, then D is a function of u alone and D' of v alone.

7 Equation (23) admits the solution $\omega = 0$, in which case the surface degenerates into a curve. Show that the general integral of the corresponding equations

(35) is $\tan \theta/2 = Ce^{\frac{u+v \cos \sigma}{\sin \sigma}}$, take for S the line $x = 0$, $y = 0$, $z = au$ and derive the equations of the transforms of S , show that the latter are surfaces of Dini (Ex 6, § 122), or a pseudosphere.

8 Show that the Backlund transforms of the surfaces of Dini and of the pseudosphere can be found without integration, and that if the pseudosphere be transformed by the transformation of Bianchi, the resulting surface may be defined by

$$x = \frac{2a \cosh u}{\cosh^2 u + v^2} (\sin v - v \cos v), \quad y = -\frac{2a \cosh u}{\cosh^2 u + v^2} (\cos v + v \sin v), \\ z = a \left(u - \frac{2 \sinh u \cosh u}{\cosh^2 u + v^2} \right),$$

Show that the lines of curvature $v = \text{const}$ lie in planes through the z axis.

9 The tangents to a family of geodesics of the elliptic or hyperbolic type on a pseudospherical surface are normal to a W surface, the relations between the radii are respectively

$$\rho_1 - \rho = a \tanh \frac{\rho_1 + \rho}{a}, \quad \rho_1 - \rho = a \coth \frac{\rho_1 + \rho}{a}$$

where a and c are constants (cf § 76).

10 Show that the linear elements of the second surfaces of center of the W surfaces of Ex 9 are reducible to the respective forms

$$ds_2^2 = \tanh^4 \frac{u}{a} du^2 + \cosh^2 \frac{u}{a} dv^2, \quad ds_2^2 = \coth^4 \frac{u}{a} du^2 + \cosh^2 \frac{u}{a} dv^2,$$

and that consequently these surfaces are applicable to surfaces of revolution whose meridians are defined by

$$r = \frac{a}{\sqrt{a^2 \kappa^2 + 1}} \sin \phi, \quad z = a \left(\log \tan \frac{\phi}{2} + \cos \phi \right), \\ r = \frac{a}{\sqrt{1 - a^2 \kappa^2}} \sin \phi, \quad z = a \left(\log \tan \frac{\phi}{2} + \cos \phi \right),$$

where κ denotes a constant.

determine the particular form of the linear element (49), and the nature as upon the surface to which the asymptotic lines on the sheets of correspond, when

$$(a) \quad \frac{\rho_1}{\rho_2} = \text{const.}, \quad (b) \quad \frac{1}{\rho_1} - \frac{1}{\rho_2} = \text{const.}$$

in a W -surface is of the type (72), the surfaces of center are applicable and to an imaginary paraboloid of revolution.

in a W -surface is of the type (72) and the linear element of the sphere (VI, 60), the curves $u + v = \text{const.}$ and $u - v = \text{const.}$ on the spherical are geodesic parallels whose orthogonal trajectories correspond to asymptotic lines on the surfaces of center, hence on each sheet there is a family such that the tangents at their points of meeting with an asymptotic line are parallel to a plane, which varies in general with the asymptotic line.

7 that the equations

$$x = aU \cos \frac{v}{a} + \int V \sin \frac{v}{a} dv, \quad y = aU \sin \frac{v}{a} - \int V \cos \frac{v}{a} dv, \\ z = \int \sqrt{1 - a^2 U'^2} du,$$

represents an arbitrary constant, define a family of applicable molding surfaces.

in the lines of curvature in one system on a surface are plane, and the second system lie on spheres which cut the surface orthogonally, the surface of Joachimsthal

spherical lines of curvature on a surface of Joachimsthal have constant curvature, the radius of geodesic curvature being the radius of the sphere the curve lies

in the lines of curvature in one system on a surface lie on concentric spheres is a surface of Monge, whose director-developable is a cone with its vertex at the center of the spheres, and conversely

the sheets of the evolute of a surface of Monge are the director-developable and the surface of Monge, which has the same director-developable and whose evolute is the evolute of the generating curve of the given surface

the lines of curvature in one system on a surface are plane and two in the second system are plane, then all in the latter system are plane

a surface with plane lines of curvature in both systems, in one of which the lines are circles, is

a surface of Joachimsthal

the locus of the orthogonal trajectories of a family of spheres with centers on a straight line, which pass through a circle on one of the spheres

the envelope of a family of spheres whose centers lie on a plane curve C and whose radii are proportional to the distances of these centers from a straight line in the plane of C

if an arbitrary curve C be drawn in a plane, and the plane be made to move so that a fixed line of it envelop an arbitrary space curve Γ and at the same time the plane be always normal to the principal normal to Γ , the curve C is the surface of Monge

22 If all the Bianchi transforms of a pseudospherical surface S are surfaces of Enneper (cf Ex 5, § 134), S is a surface of revolution

23 When ω has the value in Ex 5, § 134, the surfaces with the spherical representation (25), and with the linear element

$$ds^2 = \left(U_1 \cos \omega + \frac{U_1'}{U} \right)^2 du^2 + U_1^2 \sin^2 \omega dv^2,$$

where U_1 is an arbitrary function of u , are surfaces of Joachimsthal

24 If the lines of curvature in both systems be plane for a surface S with the same spherical representation of its lines of curvature as for a pseudospherical surface, S is a molding surface

25 If S is a pseudospherical surface with the spherical representation (25) of its lines of curvature, and the curves $v = \text{const}$ are plane, the function θ , given by

$$\sin \theta \frac{\partial^2 \omega}{\partial v^2} + \cos \theta \frac{\partial \omega}{\partial u} \frac{\partial \omega}{\partial v} + \sin \omega \frac{\partial \omega}{\partial v} = 0,$$

determines a transformation of Bianchi of S into a surface S_1 for which the lines of curvature $v = \text{const}$ are plane

26 A necessary and sufficient condition that the lines of curvature $v = \text{const}$ on a pseudospherical surface with the representation (25) of its lines of curvature be spherical is that

$$\cot \omega = V_1 + \frac{V}{\sin \omega} \frac{\partial \omega}{\partial v},$$

where V and V_1 are functions of v alone. Show that when ω is a solution of (2) and of

$$\left| \begin{array}{cc} \frac{1}{\sin^2 \omega} \frac{\partial \omega}{\partial u} & \frac{\partial}{\partial u} \left(\frac{1}{\sin \omega} \frac{\partial \omega}{\partial v} \right) \\ \frac{\partial}{\partial u} \left(\frac{1}{\sin^2 \omega} \frac{\partial \omega}{\partial u} \right) & \frac{\partial^2}{\partial u^2} \left(\frac{1}{\sin \omega} \frac{\partial \omega}{\partial v} \right) \end{array} \right| = 0,$$

the curves $v = \text{const}$ are plane or spherical, and that in the latter case V and V_1 can be found directly

27 Show that when ω is a solution of (23) and of

$$\frac{\partial \omega}{\partial v} \frac{\partial^3 \omega}{\partial u \partial v^2} - \frac{\partial^2 \omega}{c^2} \frac{c^2 \omega}{cu \partial v} + \frac{\partial \omega}{\partial u} \left(\frac{\partial \omega}{\partial v} \right)^3 = 0,$$

and $\frac{\partial}{\partial u} \left(\frac{1}{\cos \omega} \frac{\partial \omega}{\partial v} \right) \neq 0$, the lines of curvature $u = \text{const}$ are spherical on the pseudospherical surface with the spherical representation (2), and that when ω is such a function, upon the surfaces with the linear element

$$ds^2 = V^2 \left(\frac{\partial \omega}{\partial v} \right)^2 du^2 + \left(V \frac{\partial^2 \omega}{\partial v^2} + V \frac{c \omega}{\partial v} \right)^2 \frac{dv^2}{\left(\frac{\partial \omega}{\partial v} \right)^2},$$

or

$$ds^2 = \left(\sin \omega + V \frac{\partial \omega}{\partial v} \right)^2 du^2 + \left(\cos \omega + V + V \frac{\partial^2 \omega}{\partial v} \right)^2 \frac{dv^2}{\left(\frac{\partial \omega}{\partial v} \right)^2},$$

where V is a function of v alone, the curves $v = \text{const}$ are spherical, in the former case the spheres cut the surface orthogonally

CHAPTER IX

DEFORMATION OF SURFACES

Problem of Minding Surfaces of constant curvature According to § 48 two surfaces are *applicable* when a one-to-one correspondence can be established between them which is of such a nature that in the neighborhood of corresponding points corresponding figures are congruent or symmetric. It was seen in § 48 that two surfaces with the same linear element are applicable, corresponding curves on the two surfaces being in correspondence. But the fact that the linear elements of two surfaces are equal is not a sufficient condition that they are not applicable, because of this we have merely to recall the effect of a change of parameters, to say nothing of a change of parametric lines. Now we are brought to the following problem, first proposed by Minding *

Find a necessary and sufficient condition that two surfaces be applicable.

According to the second theorem of § 64 it follows that a necessary condition is that the total curvature of the two surfaces at corresponding points be the same. We shall show that this condition is also sufficient for surfaces of constant curvature.

In § 64 we found that when K is zero at all points of a surface, the surface is applicable to the plane. If the plane be referred to a system of straight lines parallel to the rectangular axes, its linear element is

$$ds^2 = dx^2 + dy^2$$

The analytical problem of the application of a developable surface upon the plane reduces to the determination of orthogonal families of geodesics such that when these curves are parametric the linear element takes the above form

* *Crelle* Vol XIX (1839) pp 371-387

Referring to the results of § 39, we see that in this case the factor tt_1 must equal unity. Consequently we must find a function θ such that the left hand members of the equations

$$e^{\theta} \left(\sqrt{E} du + \frac{F + iH}{\sqrt{E}} dv \right) = d(x + iy),$$

$$e^{-\theta} \left(\sqrt{E} du + \frac{F - iH}{\sqrt{E}} dv \right) = d(x - iy)$$

are exact differentials, in which case these equations give x and y by quadratures. Hence we must have

$$\frac{\partial}{\partial v} (e^{\theta} \sqrt{E}) = \frac{\partial}{\partial u} \left(e^{\theta} \frac{F + iH}{\sqrt{E}} \right), \quad \frac{\partial}{\partial v} (e^{-\theta} \sqrt{E}) = \frac{\partial}{\partial u} \left(e^{-\theta} \frac{F - iH}{\sqrt{E}} \right),$$

which are equivalent to

$$\frac{\partial \theta}{\partial u} = \frac{1}{H} \frac{\partial F}{\partial u} - \frac{F}{2EH} \frac{\partial E}{\partial u} - \frac{1}{2H} \frac{\partial E}{\partial v},$$

$$\frac{\partial \theta}{\partial v} = \frac{1}{2H} \frac{\partial G}{\partial u} - \frac{F}{2EH} \frac{\partial E}{\partial v}$$

From (V, 12) it is seen that these equations are consistent when $K=0$. In this case θ , and consequently x and y , can be found by quadratures.

The additive constants of integration are of such a character that if x_0, y_0 are a particular set of solutions, the most general is

$$x = x_0 \cos \alpha - y_0 \sin \alpha + a, \quad y = x_0 \sin \alpha + y_0 \cos \alpha + b,$$

where α, a, b are arbitrary constants.

In the above manner we can effect the isometric representation of any developable surface upon the plane, and consequently upon itself or any other developable. These results may be stated thus:

A developable surface is applicable to itself, or to any other developable, in a triple infinity of ways, and the complete determination of the applicability requires quadratures only.

Incidentally we have the two theorems

The geodesics upon a developable surface can be found by quadrature.

If the total curvature of a quadratic form be zero, the quadratic form is reducible by quadratures to $dx^2 + dy^2$.

is now that the total curvature of two surfaces S, S_1 is a real constant. Let P and P_1 be points on S respectively, C and C_1 geodesics through these respective points and take P and P_1 for the poles and C and C_1 for the $\theta = 0$ of a polar geodesic system on these surfaces. The elements are accordingly (VIII, 6)

$$ds^2 = du^2 + \sin^2 \frac{u}{a} dv^2, \quad ds_1^2 = du_1^2 + \sin^2 \frac{u_1}{a} dv_1^2$$

the equations $u_1 = u, \quad v_1 = \pm v$

show an isometric representation of one surface upon the other in which P and C correspond to P_1 and C_1 respectively. If we take as the upper or lower sign in the second equation corresponding figures are equal or symmetric. Similar results obtain for pseudospherical surfaces. Hence we have

Two surfaces of constant curvature, different from zero, are applicable so that a given point and geodesic through it on one correspond to a given point and geodesic through it on the other.

Consequently, a surface of constant curvature can be applied to another so that a given point shall go into any other point and a geodesic through the former into one through the latter. Combined with the last theorem of § 117, we have

Any developable surface of constant curvature can be applied to any surface of the same curvature, in a triple infinity of ways. The complete realization of the applicability requires the solution of a Riccati equation.

Solution of the problem of Minding. We proceed to the solution of a necessary and sufficient condition that two surfaces of variable curvature be applicable. Let their linear elements be

$$ds^2 = E du^2 + 2F du dv + G dv^2, \quad ds'^2 = E' du'^2 + 2F' du' dv' + G' dv'^2$$

the surfaces S and S' are applicable if there exist two independent functions

$$\phi(u, v) = \phi'(u', v'), \quad \psi(u, v) = \psi'(u', v'),$$

giving a one-to-one correspondence between the surfaces of constant curvature that by means of (1) either of the above quadratic elements can be transformed into the other.

It is evident that if the two surfaces are applicable, the differential parameters formed with respect to the two linear elements are equal. Hence a necessary condition is

$$(2) \quad \Delta_1 \phi = \Delta'_1 \phi', \quad \Delta_1(\phi, \psi) = \Delta'_1(\phi', \psi'), \quad \Delta_1 \psi = \Delta'_1 \psi',$$

where the primes indicate functions pertaining to S' . These conditions are likewise sufficient that the transformation (1) change either of the above quadratic forms into the other. For, if the curves $\phi = \text{const}$, $\psi = \text{const}$, $\phi' = \text{const}$, $\psi' = \text{const}$ be taken for the parametric curves on S and S' respectively, the respective linear elements may be written (cf § 37)

$$ds^2 = \frac{\Delta_1 \psi d\phi^2 - 2 \Delta_1(\phi, \psi) d\phi d\psi + \Delta_1 \phi d\psi^2}{\Delta_1 \phi \Delta_1 \psi - \Delta_1^2(\phi, \psi)},$$

$$ds'^2 = \frac{\Delta'_1 \psi' d\phi'^2 - 2 \Delta'_1(\phi', \psi') d\phi' d\psi' + \Delta'_1 \phi' d\psi'^2}{\Delta'_1 \phi' \Delta'_1 \psi' - \Delta_1'^2(\phi', \psi')}$$

Hence when equations (1) and (2) hold, the surfaces are applicable.

The next step is the determination of equations of the form (1). Since the curvature of two applicable surfaces at corresponding points is the same, one such equation is afforded by the necessary condition

$$(3) \quad K(u, v) = K'(u', v')$$

The first of equations (2) is

$$(4) \quad \Delta_1 K = \Delta'_1 K'$$

Both members of this equation cannot vanish identically. For, in this case the curves $K = \text{const}$ and $K' = \text{const}$ would be minimal (§ 37), and consequently imaginary. If these two equations are independent of one another, that is,

$$\Delta_1 K \neq f(K), \quad \Delta'_1 K' \neq f(K'),$$

they establish a correspondence, and the condition that it be isometric is, as seen from (2),

$$\Delta_1(K, \Delta_1 K) = \Delta'_1(K', \Delta'_1 K'), \quad \Delta_1 \Delta_1 K = \Delta'_1 \Delta'_1 K'$$

If, however,

$$(5) \quad \Delta_1 K = f(K), \quad \Delta'_1 K' = f(K'),$$

we may take for the second of (1)

$$(6) \quad \Delta_2 K = \Delta'_2 K',$$

unless

$$(7) \quad \Delta_2 K = \phi(K), \quad \Delta'_2 K' = \phi(K')$$

dition be not satisfied, the conditions that (3), (6) define the correspondence are

$$\Delta_1(K, \Delta_2 K) = \Delta'_1(K', \Delta'_2 K'), \quad \Delta_1 \Delta_2 K = \Delta'_1 \Delta'_2 K' *$$

we consider the case where both (5) and (7) hold. Since $\Delta_1 K$ and $\Delta_2 K$ is a function of K , the curves $K = \text{const.}$ orthogonal trajectories $t = \text{const.}$ form an isothermal system on S (§ 41). Moreover, the function t can be found at once, and the linear element is reducible to

$$ds^2 = \frac{1}{f(K)} (dK^2 + e^{2 \int \frac{\phi(K)}{f(K)} dK} dt^2)$$

particular $\Delta_2 K = 0$, the linear element is

$$ds^2 = \frac{1}{f(K)} (dK^2 + dt^2)$$

hence the linear element of S' is reducible to

$$ds'^2 = \frac{1}{f(K')} (dK'^2 + e^{2 \int \frac{\phi(K')}{f(K')} dK'} dt'^2),$$

particular case $\Delta'_2 K' = 0$, to

$$ds'^2 = \frac{1}{f(K')} (dK'^2 + dt'^2)$$

in the other case the equations

$$K = K', \quad t = \pm t' + a,$$

where a is an arbitrary constant, define the applicability of the

surface thus treated all possible cases and found that it can be determined without quadrature whether two surfaces are applicable. Moreover, in the first two cases the equations defining the correspondence follow directly, but in the last case the determination requires a quadrature. The last case differs also in this respect that the correspondence can be effected in an infinity of ways, whereas in the other two cases it is unique.

The surface may be referred to the curves $\sigma = \text{const.}$ and their orthogonal trajectories. The equation (6) may be replaced by $\Delta_2 \sigma = \Delta_1 \sigma$ and it can be shown that $\sigma = \Delta_1(\sigma', \Delta_2 \sigma')$ is a consequence of the other conditions. Cf. Darboux

Furthermore, we notice from (8) that in the third case the face S is applicable to a surface of revolution, the parallels latter corresponding to the curves $K = \text{const}$ of the former versely, the linear element of every surface applicable to a surface of revolution can be put in the form (8). For, a necessary and sufficient condition that a surface be applicable to a surface of revolution is that its linear element be reducible to

$$ds^2 = du^2 + U^2 dv^2,$$

where U is a function of u alone (§ 46). Now

$$\Delta_1 u = 1, \quad K = -\frac{U''}{U}, \quad \Delta_2 \int \frac{du}{U} = 0$$

From the second it follows that $u = F(K)$, and consequently $\int \frac{du}{U} = F_1(K)$. When these values are substituted in the first, we have, in consequence of Ex 5, p 91,

$$\Delta_1 K = f(K), \quad \Delta_2 K = \phi(K)$$

Hence we have the theorem

Equations (9) constitute a necessary and sufficient condition for a surface to be applicable to a surface of revolution

The equations

$$K = K, \quad t = \pm t' + a$$

define an isometric representation of a surface with the line element (8) upon itself. Therefore we have

Every surface applicable to a surface of revolution admits continuous deformation into itself in such a way that each $K = \text{const}$ slides over itself

Conversely, every surface applicable to itself in an infinity of ways is applicable to a surface of revolution. For, if the curvature is constant, the surface is applicable to a surface of revolution (§ 135), and the only case in which two surfaces of variable curvature are applicable in an infinity of ways is that for which conditions (5) and (7) are satisfied

137 Deformation of minimal surfaces These results suggest a means of determining the minimal surfaces applicable to a surface of revolution. In the first place we inquire under what conditions no minimal surfaces are applicable. The latter problem reduces to the determination of two pairs of parameters, u, v and u_1, v_1 , and two pairs of functions, $F(u), \Phi(v)^*$ and $F_1(u_1), \Phi_1(v_1)$, which satisfy the condition

$$(10) \quad (1 + uv)^2 F(u) \Phi(v) du dv = (1 + u_1 v_1)^2 F_1(u_1) \Phi_1(v_1) du_1 dv_1$$

From the nature of this equation it follows that the equations which serve to establish the correspondence between the two surfaces are either of the form

$$(1) \quad u_1 = \phi(u), \quad v_1 = \psi(v),$$

or

$$(2) \quad u_1 = \phi(v), \quad v_1 = \psi(u)$$

If either set of values for u_1 and v_1 be substituted in (10), and after removing the common factor $du dv$ we take the logarithmic derivative with respect to u and v , we obtain

$$\frac{\phi' \psi'}{(1 + u_1 v_1)^2} = \frac{1}{(1 + uv)^2}$$

as this may be written

$$(3) \quad \frac{du_1 dv_1}{(1 + u_1 v_1)} = \frac{du dv}{(1 + uv)},$$

The spherical images of corresponding parts on the two surfaces are equal or symmetric according as (11) or (12) obtains (§ 47). The latter case reduces to the former when the sense of the normal to either surface is changed. When this has been done, corresponding spherical images are equal and can be made to coincide by a rotation of the unit sphere about a diameter. Hence one surface can be so displaced in space that corresponding normals become parallel, in which case the two surfaces have the same representation, that is, $u_1 = u, v_1 = v$. Now equation (10) is

$$I(u) \Phi(v) = F_1(u) \Phi_1(v),$$

which is equivalent to

$$F_1(u) = c F(u), \quad \Phi_1(v) = \frac{1}{c} \Phi(v),$$

where c denotes a constant. If the surfaces are real, c must be of the form $e^{i\alpha}$. Hence, in consequence of § 113, we have the theorem

A minimal surface admits of a continuous deformation into an infinity of minimal surfaces, which are either associate to it or can be made such by a suitable displacement

We pass to the determination of a minimal surface which admits of a continuous deformation into itself, and consequently is applicable to a surface of revolution. In consequence of the interpretation of equation (13) it follows that if a minimal surface be deformed continuously into itself, a point p on the sphere tends to move in the direction of the small circle through p , whose axis is the momentary axis of rotation, and consequently each of these small circles moves over itself. From § 47 it follows that if the axis of rotation be taken for the z -axis, these small circles are the curves $uv = \text{const}$. In the deformation each point of the surface moves along the curve $K = \text{const}$ through it. Hence K is a function of uv . From (VII, 100, 102) we have

$$K = \frac{-4}{(1+uv)^4 F(u) \Phi(v)},$$

consequently $F(u) \Phi(v)$ must be a function of uv , and hence

$$\frac{uF'(u)}{F(u)} = \frac{v\Phi'(v)}{\Phi(v)}$$

The common value of these two terms is a constant. If it be denoted by κ , we have

$$F(u) = cu^\kappa, \quad \Phi(v) = c_1 v^\kappa,$$

where c and c_1 are constants. Hence from (VII, 98) we have

Any minimal surface applicable to a surface of revolution can be defined by equations of the form

$$(14) \quad \begin{cases} x = \frac{1}{2} c \int (1-u^2) u^\kappa du + \frac{1}{2} c_1 \int (1-v^2) v^\kappa dv, \\ y = \frac{i}{2} c \int (1+u^2) u^\kappa du - \frac{i}{2} c_1 \int (1+v^2) v^\kappa dv, \\ z = c \int u^{\kappa+1} du + c_1 \int v^{\kappa+1} dv, \end{cases}$$

where c , c_1 , and κ are arbitrary constants

the curves $K = \text{const.}$ are represented on the sphere by the circles whose axis is the z -axis, in each finite deformation of the surface into itself, as well as in a very small one, the unit sphere undergoes a rotation about this axis. In § 47 it was seen that such a rotation is equivalent to replacing u, v by $ue^{i\alpha}, ve^{-i\alpha}$, where α is the angle of rotation. Hence the continuous deformation of the surface (14) is defined by the equations resulting from the substitution in (14) of $ue^{i\alpha}, ve^{-i\alpha}$ for u, v respectively.

An important property of the surfaces (14) is discovered when we rotate such a surface to a rotation of angle α about the z -axis. Let $\bar{S}$ denote the surface in its new position, and write its equations in the form

$$\bar{x} = \frac{1}{2} \int (1 - \bar{u}^2) \bar{F}(\bar{u}) d\bar{u} + \frac{1}{2} \int (1 - \bar{v}^2) \bar{\Phi}(\bar{v}) d\bar{v},$$

analogously for $\bar{y}$ and $\bar{z}$. Between the parameters $\bar{u}, \bar{v}$ and u, v the following relations hold

$$\bar{u} = ue^{i\alpha}, \quad \bar{v} = ve^{-i\alpha},$$

have also

$$\bar{x} = x \cos \alpha - y \sin \alpha, \quad \bar{y} = x \sin \alpha + y \cos \alpha, \quad \bar{z} = z$$

Substituting these equations with (14), we find

$$\bar{F}(\bar{u}) = cu^{\kappa} e^{-i\alpha(\kappa+2)}, \quad \bar{\Phi}(\bar{v}) = c_1 v^{\kappa} e^{i\alpha(\kappa+2)}$$

or the correspondence defined by $\bar{u} = u, \bar{v} = v$, the surface $\bar{S}$ is associate of S , unless $\kappa + 2 = 0$, in which case it is the same

surface. Consider the latter case, and remark that its equations are (see § 110)

$$\left[c \left(\frac{1}{u} + u \right) \right], \quad y = R \left[ic \left(\frac{1}{u} - u \right) \right], \quad z = -2R \left[c \log u \right]$$

where u is replaced by $ue^{i\alpha}, ve^{-i\alpha}$, and the resulting expressions be denoted by x_1, y_1, z_1 , we have

$$\bar{x} = x \cos \alpha - y \sin \alpha, \quad \bar{y} = x \sin \alpha + y \cos \alpha, \quad \bar{z} = z + 2R(i\alpha c)$$

Thus, in a continuous deformation, the surface slides over itself in a helicoidal motion. Consequently it is a helicoid. Moreover, it is the only minimal helicoid. For, every helicoid is applicable

to a surface of revolution, and each minimal surface applicable to a surface of revolution with the z axis for the axis of revolution of the sphere is defined by (14). But only when $\kappa = -2$ will the substitution of $ue^{i\alpha}$, $ve^{-i\alpha}$ give a set of equations such as (15). Hence we have

The helicoidal minimal surfaces are defined by the Weierstrass formulas when $F(u) = c/u^2$

And we may state the other results thus

If any nonhelicoidal minimal surface, which is applicable to a surface of revolution, be rotated through any angle about the axis of the unit sphere whose small circles represent the curves $K = \text{const.}$ on the surface, and a correspondence with parallelism of tangent planes be established between the surfaces, they are associate, consequently the associates of such a minimal surface are superposable

EXAMPLES

- 1 Find under what conditions the surfaces, whose equations are

$$\begin{aligned} x &= r \cos v, & y &= r \sin v, & z &= f(r), \\ x_1 &= r \cos v, & y_1 &= r \sin v, & z_1 &= F(r) + av, \end{aligned}$$

can be brought into a one-to-one correspondence, so that the total curvature at corresponding points is the same. Determine under what condition the surfaces are applicable.

- 2 If the tangent planes to two applicable surfaces at corresponding points are parallel, the surfaces are associate minimal surfaces.

- 3 Show that the equations

$$x = e^{\alpha u}, \quad y = e^{-\alpha v}, \quad z = ae^{\alpha u^2} + be^{-\alpha v^2},$$

- where α is a real parameter, and a and b are constants, define a family of paraboloids which have the same total curvature at points with the same curvilinear coordinates. Are these surfaces applicable to one another?

- 4 Find the geodesics on a surface with the linear element

$$ds^2 = \frac{du^2 - 4uvdudv + 4vdv^2}{4(u-v^2)}$$

Show that the surface is applicable to a surface of revolution, and determine the form of a meridian of the latter.

- 5 Determine the values of the constants a and b in

$$ds^2 = du^2 + [(u + av)^2 + b^2] dv^2,$$

so that a surface with this linear element shall be applicable to

- (a) the right helicoid
(b) the ellipsoid of revolution

necessary and sufficient condition that a surface be applicable to a surface is that each curve of a family of geodesic parallels have constant curvature

so that the helicoidal minimal surfaces are applicable to the catenoid and the helicoid

Second general problem of deformation We have seen that it always be determined whether or not two given surfaces are applicable to one another. The solution of this problem was an important contribution to the theory of deformation. An equally important problem, but a more difficult one, is the following

termine all the surfaces applicable to a given one

This problem was proposed by the French Academy in 1859, and has been studied by the most distinguished geometers ever since. Although it has not been solved in the general case, its profound study has led to many interesting results, some of which we shall derive. Let a linear element of the given surface be

$$ds^2 = E du^2 + 2F dudv + G dv^2,$$

then a surface applicable to it is determined by this form and by a linear element $D du^2 + 2D' dudv + D'' dv^2$, whose coefficients satisfy Gauss and Codazzi equations (§ 64). Conversely, every set of functions D, D', D'' of these equations defines a surface applicable to a given one, and the determination of the Cartesian coordinates of the corresponding surface requires the solution of a Riccati equation. Neither the Codazzi equations, nor a Riccati equation, can be integrated in the general case with our present knowledge of differential equations. Later we shall make use of this method in a number of particular cases, but for the present we proceed to the solution of another means of attacking the general problem. Let the values of D, D', D'' obtained from the Gauss equations be substituted in the equation $H \Delta_1 \Delta_2 = D D'' - D'^2$, the result is reducible, in consequence of the identity $\Delta_1 \Delta_2 = 1 - \lambda$ (p. 120), to

$$III \quad (1 - \Delta_1 \Delta_2)$$

$$= \left(\frac{\partial^2 x}{\partial u^2} - \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\} \frac{\partial x}{\partial u} - \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} \frac{\partial x}{\partial v} \right) \left(\frac{\partial^2 x}{\partial v^2} - \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} \frac{\partial x}{\partial u} - \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} \frac{\partial x}{\partial v} \right) \\ - \left(\frac{\partial^2 x}{\partial u \partial v} - \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} \frac{\partial x}{\partial u} - \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} \frac{\partial x}{\partial v} \right)$$

This equation, which is satisfied also by y and z , involves only E, F, G and their derivatives, and consequently its integration will give the complete solution of the problem. It is linear in $\left[\frac{\partial^2 x}{\partial u^2} \frac{\partial^2 x}{\partial v^2} - \left(\frac{\partial^2 x}{\partial u \partial v} \right)^2 \right]$, $\frac{\partial^2 x}{\partial u^2}$, $\frac{\partial^2 x}{\partial u \partial v}$, $\frac{\partial^2 x}{\partial v^2}$, and therefore is of the form studied by Ampère. Hence we have the theorem

The determination of all surfaces applicable to a given one requires the integration of a partial differential equation of the second order of the Ampère type

In consequence of (16) and (V, 36) we have that the coordinates of a surface with the linear element

$$(17) \quad ds^2 = E du^2 + 2F dudv + G dv^2$$

are integrals of

$$(18) \quad \Delta \theta = (1 - \Delta_1 \theta) K,$$

Differential parameters being formed with respect to (17). We find that when one of these coordinates is known the other two can be found by quadratures.

Our general problem may be stated thus

Given three functions E, F, G of u and v , to find all functions x, y, z of u and v which satisfy the equation

$$dx^2 + dy^2 + dz^2 = E du^2 + 2F dudv + G dv^2,$$

where du and dv may be chosen arbitrarily

Darboux* observed that as the equation may be written

$$(19) \quad dx^2 + dy^2 = E du^2 + 2F dudv + G dv^2 - dz^2,$$

whose left-hand member is the linear element of the plane, or of a developable surface, the total curvature of the quadratic form

$$(20) \quad \left[E - \left(\frac{\partial z}{\partial u} \right)^2 \right] du^2 + 2 \left[F - \frac{\partial z}{\partial u} \frac{\partial z}{\partial v} \right] dudv + \left[G - \left(\frac{\partial z}{\partial v} \right)^2 \right] dv^2$$

must be zero (§ 64)

In order to find the condition for this, we assume that z is known, and take for parametric lines the curves $z = \text{const}$ and their

nal trajectories for $v = \text{const}$. With this choice of parameters the right-hand member of (19) reduces to $(E-1)dz^2 + Gdv^2$. In addition that the curvature of this form be zero is

$$(E-1) \left[\frac{\partial E}{\partial v} \frac{\partial G}{\partial v} + \left(\frac{\partial G}{\partial z} \right)^2 \right] + G \left[\frac{\partial E}{\partial z} \frac{\partial G}{\partial z} + \left(\frac{\partial E}{\partial v} \right)^2 \right] \\ - 2(E-1)G \left[\frac{\partial^2 E}{\partial v^2} + \frac{\partial^2 G}{\partial z^2} \right] = 0, \\ \frac{\partial E}{\partial z} \frac{\partial G}{\partial z} + \left(\frac{\partial E}{\partial v} \right)^2 + 4E^2 G(E-1)K = 0,$$

K denotes the curvature of the surface. But this is the condition also that z be a solution of (18) when the differential operators are formed with respect to $E dz^2 + G dv^2$. However, the parameters of equation (18) are differential parameters, consequently the solution of this equation whatever be the parametric curves reversing the above steps we prove the theorem

If z is any integral of the equation (18), the quadratic form (20) is curvature

When such a solution is known we can find by quadratures (5) two functions x, y such that the quadratic form (20) is $dx^2 + dy^2$, provided that

$$\left[E - \left(\frac{\partial z}{\partial u} \right)^2 \right] \left[G - \left(\frac{\partial z}{\partial v} \right)^2 \right] - \left[F - \frac{\partial z}{\partial u} \frac{\partial z}{\partial v} \right]^2 > 0,$$

$\Delta_1 z < 1$. Hence we have the theorem

If a solution of $\Delta_{22}\theta = (1 - \Delta_1\theta)K$ such that $\Delta_1 z < 1$, it is one rectangular coordinates of a surface with the given linear element and the other two coordinates can be obtained by quadratures

Deformations which change a curve on the surface into a curve in space. We consider the problem

A surface be deformed in such a manner that a given curve C comes into coincidence with a given curve Γ in space?

The surface be referred to a family of curves orthogonal to C their orthogonal trajectories, C being the curve $v = 0$, and u being the parameter u , so that $E = 1$ for $v = 0$. The same conclusions hold for Γ on the deform

Since the geodesic curvature of C is unaltered in the deformation (§ 58), it follows from the equation (IV, 47) for the new surface that

$$(21) \quad \rho = \rho_g \sin \bar{\omega},$$

that the deformation is impossible, if the curvature of Γ at a point is less than the geodesic curvature of C at the corresponding point. Since both ρ and ρ_g are known, equation (21) determines $\bar{\omega}$, and consequently the direction of the normal to the new surface along Γ is fixed. This being the case, the direction of the tangent to the curves $u = \text{const.}$ on the new surface at points of Γ is found, and so we have the values of $\frac{1}{\sqrt{G}} \frac{\partial x}{\partial v}$, $\frac{1}{\sqrt{G}} \frac{\partial y}{\partial v}$, $\frac{1}{\sqrt{G}} \frac{\partial z}{\partial v}$ for as well as $\frac{\partial x}{\partial u}$, $\frac{\partial y}{\partial u}$, $\frac{\partial z}{\partial u}$ for $v = 0$, the latter being the direction-cosines of the tangent to Γ . If these expressions be differentiated

respect to u , we obtain the values of $\frac{\partial^2 x}{\partial u^2}$, $\frac{\partial^2 y}{\partial u^2}$, $\frac{\partial^2 z}{\partial u^2}$, $\frac{\partial^2 x}{\partial u \partial v}$, $\frac{\partial^2 y}{\partial u \partial v}$, for $v = 0$. Since $F = 0$ and $E = 1$ for $v = 0$, the Gauss equation (V, 7) for $v = 0$ are

$$(22) \quad \begin{cases} \frac{\partial^2 x}{\partial u^2} = -\frac{1}{2G} \frac{\partial E}{\partial v} \frac{\partial x}{\partial v} + DX, \\ \frac{\partial^2 x}{\partial u \partial v} = \frac{1}{2} \frac{\partial E}{\partial v} \frac{\partial x}{\partial u} + \frac{1}{2G} \frac{\partial G}{\partial u} \frac{\partial x}{\partial v} + D'X, \\ \frac{\partial^2 x}{\partial v^2} = -\frac{1}{2} \frac{\partial G}{\partial u} \frac{\partial x}{\partial u} + \frac{1}{2G} \frac{\partial G}{\partial v} \frac{\partial x}{\partial v} + D''X \end{cases}$$

All the terms of the first two equations have been determined except D and D' , hence the latter are given by these equations. Since the total curvature K is unaltered by the deformation, it is known at all points of Γ , consequently D'' is given by $H(DD'' - D')$, unless D is zero, in which case Γ is an asymptotic curve and $\rho = \rho_g$. When D'' is found we can obtain the value of $\frac{\partial^2 x}{\partial v^2}$ from the last of equations (22). From the method of derivation of equation (16) it follows that the above process is equivalent to finding the value of $\frac{\partial^2 x}{\partial v^2}$ from equation (16), which is possible unless D is zero. Excluding this exceptional case, we remark that if equations

differentiated with respect to u , we obtain the values of all the derivatives of x of the third order for $v = 0$ except $\frac{\partial^2 x}{\partial v^2}$. The latter may be obtained from the equation which results from the differentiation of equation (16) with respect to v . By continuing this process we obtain the values for $v = 0$ of the derivatives of x of all orders, and likewise of y and z . If we indicate by subscript null the values of functions, when $u = u_0, v = 0$, the expansions

$$\beta) x = x_0 + \left(\frac{\partial x}{\partial u}\right)_0 u + \left(\frac{\partial x}{\partial v}\right)_0 v + \frac{1}{2} \left(\frac{\partial^2 x}{\partial u^2}\right)_0 u^2 + \left(\frac{\partial^2 x}{\partial u \partial v}\right)_0 uv + \frac{1}{2} \left(\frac{\partial^2 x}{\partial v^2}\right)_0 v^2 + \dots,$$

and similar expansions for y and z , are convergent in general, as Poincaré has shown,* and x, y, z thus defined are the solutions of equation (16) which for $v = 0$ satisfy the given conditions. Hence

A surface S can be deformed in such a manner that a curve C upon it comes into coincidence with a given curve Γ , provided that the curvature of Γ at each point is greater than the geodesic curvature of C at the corresponding point.

There remains the exceptional case $\rho = \rho_g$. If the desired deformation is possible, Γ is an asymptotic line on the deformant, and consequently, by Enneper's theorem (§ 59), its radius of torsion must satisfy the condition $\tau^2 = -1/K$. Hence when C is given, Γ is determined, if it is to be an asymptotic line.

If Γ satisfies these conditions, the value of D'' for $v = 0$ is arbitrary, as we have seen. But when it has been chosen, the further determination of the values of the derivatives of x, y, z of higher order for $v = 0$ is unique, it being the same as that pursued in the general case. Hence equation (16) admits as solution a family of these surfaces, depending upon an arbitrary function. For all these surfaces the directions of the tangent planes at each point of Γ are the same. Hence we have the theorem

Given a curve C upon a surface S , there exists in space a unique curve Γ with which C can be brought into coincidence by a deformation of S in an infinity of ways, moreover, all the new surfaces are tangent to one another along Γ .

* Cf. Goursat, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre*, chap. II. Paris, 1896.

If C is an asymptotic line on S , it may be taken for Γ , hence
A surface may be subjected to a continuous deformation during which a given asymptotic line is unaltered in form and continues to be an asymptotic line on each deform

This result suggests the problem

Can a surface be subjected to a continuous deformation in which a curve other than an asymptotic line is unaltered?

By hypothesis the curvature is not changed and the geodesic curvature is necessarily invariant, hence from (21) we have that $\kappa \bar{\omega}$ must have the same value for all the surfaces. If $\bar{\omega}$ is the same for all surfaces, the tangent plane is the same, and consequently the expansions (23) are the same. Hence all the surfaces coincide in this case. However, there are always two values of $\bar{\omega}$ for which $\kappa \bar{\omega}$ has the same value, unless $\bar{\omega}$ is a right angle. Hence it is possible to have two applicable surfaces passing through a curve whose points are self-correspondent, but not an infinity of such surfaces. Therefore

An asymptotic line is the only curve on a surface which can remain unaltered in a continuous deformation

140 Lines of curvature in correspondence We inquire whether surface S can be deformed in such a manner that a given curve upon it may become a line of curvature on the new surface. Suppose it is possible, and let Γ denote this line of curvature. The radii of curvature and torsion of Γ must satisfy (21) and $1/\tau - d\bar{\omega}/ds = 0$ (cf § 59), where ρ_v is the same for Γ as for C . If we choose for $\bar{\omega}$ any function whatever, the functions ρ and τ are thus determined, and Γ is unique. Since ω fixes the direction of the tangent plane to the new surface along Γ , there is only one deform S of the kind desired for each choice of ω (cf § 139). Hence

A surface can be deformed in an infinity of ways so that a given curve upon it becomes a line of curvature on the deform

This result suggests the following problem of Bonnet*

To determine the surfaces which can be deformed with preservation of their lines of curvature

* Memoire sur la théorie des surfaces applicables sur une surface donnée, *Journal de l'école Polytechnique*, Cahier 42 (1867), p. 58

the method of Bonnet in making use of the formulas in the form (V, 48, 55) We assume that the lines are parametric In this case these equations reduce to

$$\begin{aligned} &= -qr_1, \quad \frac{\partial q}{\partial v} = rp_1, \quad \frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u} = -qp_1, \\ &= -\frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v}, \quad r_1 = \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \end{aligned}$$

It follows that if S and S' are two applicable surfaces to corresponding lines of curvature, the functions have the same value for both surfaces, and consequently the value of the product qp_1 . Hence our problem reduces to the determination of two sets of functions p_1, q, p'_1, q' , satisfying the above equations. In consequence of the identity

$$p'_1 q' = p_1 q,$$

the first two of (24)

$$p'_1 \frac{\partial p'_1}{\partial u} = p_1 \frac{\partial p_1}{\partial u}, \quad q' \frac{\partial q'}{\partial v} = q \frac{\partial q}{\partial v},$$

the integrals are $p_1'^2 = p_1^2 + f(v)$, $q'^2 = q^2 + \phi(u)$, where $f(v)$ and $\phi(u)$ are functions of v and u respectively. The parameters u, v are chosen so that these functions become constants α, β , respectively

$$p_1'^2 = p_1^2 + \alpha, \quad q'^2 = q^2 + \beta$$

When these equations are multiplied together, the resulting equation is transformed by means of (25) to either of the forms

$$p_1'^2 \beta + q'^2 \alpha + \alpha \beta = 0, \quad p_1' \beta + q'^2 \alpha - \alpha \beta = 0$$

From the first we see that α and β cannot both be positive if S is real, and from the second that they cannot both be negative. We assume α and β positive, and without loss of generality write

$$p_1'^2 = p_1^2 + 1, \quad q'^2 = q^2 + 1$$

Equation (28) reduces to $p_1^2 - q^2 = 1$. In conformity with this we put p_1 as a function of ω , thus

$$p_1 = \cosh \omega, \quad q = \sinh \omega$$

Equation (29) may be replaced by

$$p_1' = \sinh \omega, \quad q' = \cosh \omega$$

Moreover, the fundamental equations (24) reduce to

$$\frac{\partial \omega}{\partial v} = r = -\frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v}, \quad \frac{\partial \omega}{\partial u} = -r_1 = -\frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u},$$

$$\frac{\partial^2 \omega}{\partial u^2} + \frac{\partial^2 \omega}{\partial v^2} = -\sinh \omega \cosh \omega$$

Comparing these results with § 118, we see that the spherical representation of lines of curvature of the surfaces S and S' relatively is the same as of the lines of curvature of a spherical surface and of its Hazzidakis transform. Conversely, we have that a surface of this kind admits of an applicable surface with lines of curvature in correspondence.

The preceding investigation rested on the hypothesis that neither the first nor second of equations (24) vanishes identically. Suppose that the second vanishes, then q is a function of u alone, say $q = \phi(u)$. Since the product $p_1 q$ differs from the total curvature only by a factor (cf § 70), p_1 cannot be zero, therefore $r = 0$ and $q' = \phi'(u)$. Equation (25) is now of the form $p_1 \phi(u) = p_1' \phi_1(u)$. If p_1' be eliminated from this equation and the first of (27), it is found that p_1 also is a function of u alone. Hence the curves $v = \text{const}$ on the sphere are great circles with a common diameter, and therefore the surface is a *molding surface* (§ 130). The parameter u may be chosen that we may take $q = 1$ and $p_1 = U$, then from (27) and (25) find $p_1' = \sqrt{U^2 + \alpha}$, $q' = U/\sqrt{U^2 + \alpha}$, where α is an arbitrary constant. Hence we have the theorem:

A necessary and sufficient condition that a surface admit of an applicable surface with lines of curvature in correspondence is that the surface have the same spherical representation of its lines of curvature as a spherical surface Σ , or be a molding surface. In the first case there is one applicable surface, and the spherical representation of its lines of curvature is the same as of the Hazzidakis transform of Σ , in the second case there is an infinity of applicable surfaces.

141 Conjugate systems in correspondence. When two surfaces are applicable to one another, there is a system of corresponding lines which is conjugate for both surfaces (cf § 56). The results of § 140 show that for a given conjugate system on a surface

there is not in general a surface S_1 applicable to S with the corresponding system conjugate. We inquire under what conditions a given conjugate system of S possesses this property.

Let S be referred to the given conjugate system. If the corresponding system on an applicable surface S_1 is conjugate, we have

$$D' = D'_1 = 0, \quad D_1 D''_1 = D D'',$$

for the total curvature of the two surfaces is the same. We replace this equation by the two

$$D_1 = \tanh \theta \cdot D, \quad D''_1 = \coth \theta \cdot D'',$$

thus defining a function θ . The Codazzi equations for S are

$$\frac{\partial D}{\partial v} - \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} D + \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} D'' = 0, \quad \frac{\partial D''}{\partial u} + \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} D - \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} D'' = 0$$

Since these equations must be satisfied by D_1 and D''_1 , we have

$$(30) \quad \frac{\partial \theta}{\partial u} = - \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} \frac{D}{D''} \tanh \theta, \quad \frac{\partial \theta}{\partial v} = - \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} \frac{D''}{D} \coth \theta$$

The condition of integrability of (30) is reducible to

$$(31) \quad \cosh \theta \left[\frac{\partial}{\partial u} \left(\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} \frac{D''}{D} \right) - \frac{\partial}{\partial v} \left(\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} \frac{D}{D''} \right) \right] \\ + \left[\frac{\partial}{\partial v} \left(\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} \frac{D}{D''} \right) + 2 \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} \right] = 0$$

As the two roots of this equation differ only in sign, and thus lead to symmetric surfaces, we need consider only one. If it be substituted in (30) we obtain two conditions upon h , k , G , D , D'' , which are necessary in order that S admit of an applicable surface of the kind sought. Hence in general there is no solution of the problem. However, if the two expressions in the brackets of (31) vanish identically, the conditions of integrability of equations (30) are completely satisfied, and S admits of an infinity of applicable surfaces upon which the coordinate curves form a conjugate system. Consequently we have the theorem

If a conjugate system on a surface S corresponds to a conjugate system on more than one surface applicable to S , it corresponds to a conjugate system on an infinity of surfaces applicable to S .

We shall give this result another interpretation by considering the spherical representation of S . From (VI, 38) we have

$$(32) \quad \left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\} \frac{D}{D''} = - \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}', \quad \left\{ \begin{smallmatrix} 11 \\ 2 \end{smallmatrix} \right\} \frac{D''}{D} = - \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}',$$

where the symbols $\left\{ \begin{smallmatrix} rs \\ t \end{smallmatrix} \right\}'$ are formed with respect to the linear element of the spherical representation of S . If we substitute these values in (30), we get

$$\frac{\partial \theta}{\partial u} = \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \tanh \theta, \quad \frac{\partial \theta}{\partial v} = \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \coth \theta,$$

and the condition that these equations have an integral involving a parameter becomes

$$(33) \quad \frac{\partial}{\partial u} \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' = \frac{\partial}{\partial v} \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' = 2 \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}'$$

The first of these equations is the condition that the curves upon the sphere represent the asymptotic lines upon a certain face Σ (cf § 78). Moreover, if K denotes the total curvature Σ , and we put $K = -1/\rho^2$, we have

$$(34) \quad \frac{\partial \log \rho}{\partial u} = -2 \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}', \quad \frac{\partial \log \rho}{\partial v} = -2 \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}'$$

Now equations (33) are equivalent to (34), and

$$\frac{\partial^2 \log \rho}{\partial u \partial v} + \frac{\partial \log \rho}{\partial u} \frac{\partial \log \rho}{\partial v} = 0,$$

which reduces to $\frac{\partial^2 \rho}{\partial u \partial v} = 0$. As the general integral of this equation is $\rho = \phi(u) + \psi(v)$, where ϕ and ψ are arbitrary functions of u and v respectively, we have the following theorem due to Bianchi*:

A necessary and sufficient condition that a surface S admit a continuous deformation in which a conjugate system remains conjugate is that the spherical representation of this system be that of the asymptotic lines of a surface whose total curvature, expressed in terms of parameters referring to these lines, is of the form

$$(35) \quad K = \frac{-1}{[\phi(u) + \psi(v)]^2}$$

* *Annali Ser. 2 Vol. XVIII (1890) p. 320 also *Lezioni* Vol. II p. 83*

The pseudospherical surfaces afford an example of surfaces with of this form. In this case ϕ and ψ are constants, so that equations (34) reduce to $\left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' = \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' = 0$, which, in consequence of 2), are equivalent to $\left\{ \begin{smallmatrix} 11 \\ 2 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\} = 0$. But these are the conditions that the parametric curves on S be geodesics. A surface with conjugate system of geodesics is called a *surface of Voss*. We state these results thus

A surface of Voss admits of a continuous deformation in which the odessic conjugate system is preserved, consequently all the new surfaces are of the same kind

EXAMPLES

1 Show that every integral of the equation $\Delta_1 \theta = 1$ is an integral of the fundamental equation (18)

2 On a right helicoid the helices are asymptotic lines. Find the surfaces applicable to the helicoid in such a way that one of the helices is unaltered in form and continues to be an asymptotic line.

3 A surface applicable to a surface of revolution with the lines of curvature the two surfaces in correspondence is a surface of revolution.

4 Show that the equations

$$x = \kappa r \cos \frac{v}{\kappa} \quad y = \kappa r \sin \frac{v}{\kappa} \quad z = \int \sqrt{1 - \kappa^2 r'^2} du,$$

define a family of applicable surfaces of revolution with lines of curvature in correspondence. Discuss the effect of a variation of the parameter κ .

5 Let S denote a surface parallel to a spherical surface Σ . Find the surface applicable to S with preservation of the lines of curvature.

6 If S_1 and S be applicable surfaces referred to the common conjugate system, then coordinates x_1, y_1, z_1, x, y, z are solutions of the same point equation VI, 2b) and the function $x_1 + y_1 + z_1 - (x_2 + y_2 + z_2)$ also is a solution.

7 Show that the locus of a point which divides in constant ratio the join of corresponding points on the surfaces S_1 and S of Ex. 6 is a surface upon which the parametric lines form a conjugate system. Under what condition is this surface applicable to S_1 and S ?

8 The tetrahedral surface

$$x = \frac{1}{2}(a + u)^{\frac{2}{3}}(a + v)^{\frac{2}{3}} \quad y = \frac{1}{2}B(b + u)^{\frac{2}{3}}(b + v)^{\frac{2}{3}} \quad z = C'(c + u)^{\frac{2}{3}}(c + v)^{\frac{2}{3}},$$

admits of an infinity of deforms

$$= \frac{1}{2}A_1(a_1 + u)^{\frac{2}{3}}(a_1 + v)^{\frac{2}{3}} \quad y_1 = \frac{1}{2}B_1(b_1 + u)^{\frac{2}{3}}(b_1 + v)^{\frac{2}{3}} \quad z_1 = \frac{1}{2}C_1(c_1 + u)^{\frac{2}{3}}(c_1 + v)^{\frac{2}{3}}$$

the curves $u = v$ upon these surfaces are congruent, and consequently each is an asymptotic line on the surface through it.

the five conditions (39), (41) From this it follows that there is an arbitrary function of v involved in the problem, and consequently there is an infinity of ruled surfaces with the linear element (40).

There are two general ways in which the choice of this arbitrary function may be made, — either as determining the form of the director-cone of the required surface, or by a property of the directrix We consider these two cases

143 Method of Minding The first case was studied by Minding * He took l, m, n in the form

$$(42) \quad l = \cos \phi \cos \psi, \quad m = \cos \phi \sin \psi, \quad n = \sin \phi,$$

which evidently satisfy the second of (39) The first of (41) reduces to

$$(43) \quad \phi'^2 + \psi'^2 \cos^2 \phi = a^2$$

If we solve equations (39) and (41) for x'_0, y'_0, z'_0 , the resulting expressions are reducible by means of (VII, 63) to

$$(44) \quad x'_0 = l \cos \theta_0 + \frac{1}{a^2} [l'b \pm (mn' - m'n) \sqrt{a^2 \sin^2 \theta_0 - b^2}],$$

and analogous expressions for y'_0 and z'_0 Hence, if ϕ be an arbitrary function of v , and ψ be given by

$$(45) \quad \psi = \int \frac{\sqrt{a^2 - \phi'^2}}{\cos \phi} dv,$$

the functions x_0, y_0, z_0 , obtained from (44) by quadratures, together with l, m, n from (42), determine a ruled surface with the linear element (40)

Each choice of ϕ gives a different director-cone, which is determined by the curve in which the cone cuts the unit sphere, whose center is at the vertex of the cone Such a curve is defined by a relation $f(\phi, \psi) = 0$, so that instead of choosing ϕ arbitrarily we may take f as arbitrary, for, by combining equations (43) and $f(\phi, \psi) = 0$, we obtain the expressions for ϕ and ψ as functions of v Hence

A ruled surface may be deformed in such a way that the director cone takes an arbitrary form

on the given ruled surface is nondevelopable, the radicand is different from zero, and consequently there are two diffeomorphisms of functions x_0, y_0, z_0 . Hence there are two applicable surfaces with the same director-cone. If the parameters of transformation of these two surfaces be calculated by (VII, 73), they are found to differ only in sign. Hence we have the theorem of

led surface admits of an applicable ruled surface such that corresponding generators are parallel, and the parameters of distribution differ only in sign

Particular deformations of ruled surfaces By means of the foregoing results we prove the theorem

ed surface may be deformed in an infinity of ways so that a curve becomes plane

given curve be taken for the directrix of the original

Assuming that a deformation of the kind desired exists, we consider the plane for the xy -plane. From (44) we have

$$a^2 n \cos \theta_0 + b n' \pm (l m' - l' m) \sqrt{a^2 \sin^2 \theta_0 - b^2} = 0,$$

in consequence of (42) and (43), reduces to

$$\phi' + a^2 \sin \phi \cos \theta_0 \pm \cos \phi \sqrt{a^2 - \phi'^2} \sqrt{a^2 \sin^2 \theta_0 - b^2} = 0$$

integral of this equation involves an arbitrary constant, and the theorem is proved.

The preceding example belongs to the class of problems whose statement is as follows

form a ruled surface into a ruled surface in such a way that the image of a given curve C on the original surface shall possess a prescribed property on the resulting surface

Consider this general problem. Let the deformation be the deformation of the required surface, and let $\alpha_0, \beta_0, \gamma_0, l_0, m_0, n_0, \lambda_0, \mu_0, \nu_0$ be the direction-cosines of its tangent, principal normal, and binormal. If σ denotes the angle between the osculating plane to the curve and the tangent plane to the surface, we have

$$l = \alpha_0 \cos \theta_0 + \sin \theta_0 (l_0 \cos \sigma + \lambda_0 \sin \sigma),$$

and similar expressions for m and n . When these values are substituted in the first two of equations (41), the resulting equations are reducible, by means of the Frenet formulas (I, 50), to

$$(47) \quad \begin{cases} \frac{\cos \sigma}{\rho} = -\left(\theta'_0 + \frac{b}{\sin \theta_0}\right), \\ \left[\frac{\cos \theta_0}{\rho} + (\cos \sigma \sin \theta_0)' + \frac{\sin \sigma \sin \theta_0}{\tau}\right]^2 \\ \quad + \left[(\sin \sigma \sin \theta_0)' - \frac{\cos \sigma \sin \theta_0}{\tau}\right]^2 = a^2 - b^2 \end{cases}$$

These are two equations of condition on σ, ρ, τ , as functions of v . Each set of solutions determines a solution of the problem, for, the directrix is determined by expressions for ρ and τ , and equations (46) give the direction-cosines of the generators.

We leave it to the reader to prove the above theorem by this means, and we proceed to the proof of the theorem.

A ruled surface may be deformed in such a manner that a given curve C becomes an asymptotic line on the new ruled surface.

On the deform we must have $\sigma = 0$ or $\sigma = \pi$, so that from (47)

$$\frac{1}{\rho} = \pm \left(\theta'_0 + \frac{b}{\sin \theta_0}\right),$$

the sign being fixed by the fact that ρ is necessarily positive. The second of (47) reduces to

$$\frac{1}{\tau} = \pm \frac{\sqrt{a^2 \sin^2 \theta_0 - b^2}}{\sin^2 \theta_0}$$

If the curve with these intrinsic equations be constructed, and in the osculating plane at each point the line be drawn which makes the angle θ_0 with the tangent, the locus of these lines is a ruled surface satisfying the given conditions.

When the curve C is an orthogonal trajectory of the generators, the same is true of its deform. Hence

A ruled surface may be deformed in such a way that all the generators become the principal normals of the deform of any one of their orthogonal trajectories.

Having thus considered the deformation of ruled surfaces in which the generators remain straight, we inquire whether two

urfaces are applicable with the generators of each corresponding to curves on the other. Assume that it is possible, and let $u = \text{const}$ be the generators of S and $v = \text{const}$ the curves corresponding to the generators of S_1 . From (V, 13) it follows that the conditions for this are respectively

$$\frac{\partial}{\partial u} \log \rho = 2 \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}, \quad \frac{\partial}{\partial v} \log \rho = 2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\},$$

$K = -1/\rho^2$. But equations (48) are the necessary and sufficient conditions that there be a surface Σ applicable to S upon which the asymptotic lines are parametric (cf VI, 3). The curves $v = \text{const}$ and $u = \text{const}$ are geodesics on S and S_1 , consequently on Σ . Therefore Σ is doubly ruled. Hence

Two ruled surfaces S and S_1 are applicable to one another, the corresponding generators of S and S_1 corresponding to the two different families of generators of the quadric.

EXAMPLES

A ruled surface can be deformed into another ruled surface in such a way that a geodesic becomes a straight line.

A ruled surface formed by the binormals of a curve C can be deformed into a helicoid, the latter is the right helicoid when the torsion of C is constant. The converse also holds.

Let the hyperboloid of revolution, defined by

$$\frac{x}{c} = \frac{u}{\Delta} \cos \frac{v}{c} + \sin \frac{v}{c}, \quad \frac{y}{c} = \frac{u}{\Delta} \sin \frac{v}{c} - \cos \frac{v}{c}, \quad \frac{z}{d} = \frac{u}{\Delta},$$

where $c^2 + d^2 = \Delta^2$, the circle of gorge is a geodesic, which is met by the generators at an angle $\cos^{-1} c/\Delta$.

Now that the ruled surface which results from the deformation of the hyperboloid of Ex. 3, in which the circle of gorge becomes straight, is given by

$$x = \frac{ud}{\Delta} \cos \frac{v}{d}, \quad y = \frac{ud}{\Delta} \sin \frac{v}{d}, \quad z = \frac{uc}{\Delta} + v$$

Now that the ruled surface to which the hyperboloid of Ex. 3 is applicable by helical motion of corresponding generators is the helicoid

$$\frac{x}{c} \cos \frac{v}{c} + \frac{c^2 - d^2}{c^2 + d^2} \sin \frac{v}{c} = \frac{u}{\Delta} \sin \frac{v}{c} - \frac{c^2 - d^2}{c^2 + d^2} \cos \frac{v}{c}, \quad \frac{z}{d} = \frac{u}{\Delta} + \frac{2c}{\Delta^2} v,$$

the circle of gorge of the former corresponds to a helix upon the latter. When the directrix is a geodesic, equations (47) reduce to

$$\sin \theta_0 \cdot \theta_0 + b = 0, \quad \left(\frac{\cos \theta_0}{\rho} \pm \frac{\sin \theta_0}{r} \right)^2 = \alpha^2 - \theta_0^2$$

7 When an hyperboloid of revolution of one sheet is deformed into another ruled surface, the circle of gorge becomes a Bertrand curve and the generators are parallel to the corresponding binormals of the conjugate Bertrand curve

8 A ruled surface can be deformed in such a way that a given curve is made to lie upon a sphere of arbitrary radius

9 When a ruled surface admits a continuous deformation into itself the total curvature of the surface is constant along the line of striction, the generators meet the latter under constant angle, and the parameter of distribution is constant (cf § 126)

10 Two applicable ruled surfaces whose corresponding generators are parallel cannot be obtained from one another by a continuous deformation

GENERAL EXAMPLES

1 Determine the systems of coördinate lines in the plane such that the linear element of the plane is

$$ds^2 = \frac{du^2 + dv^2}{(U + V)^2},$$

where U and V are functions of u and v respectively

2 Solve for the sphere the problem similar to Ex 1

3 Determine the functions $\phi(u)$ and $\psi(v)$ so that the helicoids, defined by

$$x = a\sqrt{U^2 - b^2} \cos \frac{v - \phi}{a}, \quad y = a\sqrt{U^2 - b^2} \sin \frac{v - \phi}{a}, \quad z = bv + \psi,$$

shall be applicable to the surface whose equations are

$$x = U \cos v, \quad y = U \sin v, \quad z = \int \sqrt{1 - U^2} du,$$

where U is any function of u

4 Apply the method of Ex 3 to find helicoids applicable to the pseudosphere, to the catenoid

5 The equations

$$x = a\sqrt{2u - 2} \cos \frac{v}{a}, \quad y = a\sqrt{2u - 2} \sin \frac{v}{a}, \quad z = \frac{a}{2}(u - 1)$$

define a paraboloid of revolution. Show that surfaces applicable to it are defined by

$$x = \frac{ia}{2} \left[f_3 \phi_2 - f_2 \phi_3 + \int (f_2 df_3 - f_3 df_2) - \int (\phi_2 d\phi_3 - \phi_3 d\phi_2) \right],$$

$$y = \frac{ia}{2} \left[f_1 \phi_3 - f_3 \phi_1 + \int (f_3 df_1 - f_1 df_3) - \int (\phi_3 d\phi_1 - \phi_1 d\phi_3) \right],$$

$$z = \frac{ia}{2} \left[f_2 \phi_1 - f_1 \phi_2 + \int (f_1 df_2 - f_2 df_1) - \int (\phi_1 d\phi_2 - \phi_2 d\phi_1) \right]$$

where a is a real constant, and the f 's and ϕ 's are functions of the parameter u and β respectively such that

$$f_1^2 + f_2^2 + f_3^2 = 1, \quad \phi_1^2 + \phi_2^2 + \phi_3^2 = 1, \quad f_1 \phi_1 + f_2 \phi_2 + f_3 \phi_3 = u$$

6 Investigate the special case of Ex 5 for which α and β are conjugate imaginary functions, and

$$f_1 = \frac{2 + \alpha - 2\alpha^2}{2\sqrt{2}\alpha}, \quad f_2 = i \frac{2 - \alpha - 2\alpha^2}{2\sqrt{2}\alpha}, \quad f_3 = \alpha,$$

and the ϕ 's are functions conjugate imaginary to the f 's

Now that the surface of translation

$$x = a(\cos u + \cos v), \quad y = a(\sin u + \sin v), \quad z = c(u + v)$$

is applicable to a surface of revolution

Now that the minimal surfaces applicable to a spiral surface (Ex 22, p 151) are defined by the functions $F(u) = cu^m + m$, $\Phi(v) = c_1 v^m - m$, and that the associated surfaces are similar to the given one

the coefficients E, F, G of the linear element of a surface are homogeneous of u and v of order -2 , the surface is applicable to a surface of revolution

x, y, z are the coordinates of a surface S referred to a conjugate system, then

$$P \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} = P \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} = P \frac{\partial z}{\partial u}, \quad \frac{\partial x}{\partial v} = Q \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} = Q \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} = Q \frac{\partial z}{\partial v}$$

able if P and Q satisfy the conditions

$$\frac{\partial P}{\partial v} + (P - Q) \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} = 0, \quad \frac{\partial Q}{\partial u} + (Q - P) \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} = 0,$$

Christoffel symbols are formed with respect to the linear element of S on the surface S' , whose coordinates are x', y', z' , the parametric curves of a conjugate system, and that the normals to S and S' at corresponding points are equal

Now that for the surface

$$\lambda f_1(u) du + \phi_1(v), \quad y = \int \lambda f_2(u) du + \phi_2(v), \quad z = \int \lambda f_3(u) du + \phi_3(v),$$

if λ is any function of u and v , and $f_1, f_2, f_3, \phi_1, \phi_2, \phi_3$ are functions of u and v respectively, the parametric curves form a conjugate system. Apply the results to this surface, and discuss the case for which λ is independent of v

S and S_1 are two applicable surfaces, and S_1 denotes the surface corresponding to S in the same manner as S to S in Ex 10 and by means of the same P and Q , then S' and S_1 are applicable surfaces

x, y, z and x_1, y_1, z_1 are the coordinates of a pair of applicable surfaces. A second pair of applicable surfaces S' and S_1 is defined by

$$\begin{aligned} x &= x + h(z + z_1) - k(y + y_1), & x_1 &= x_1 - h(z + z_1) + k(y + y_1), \\ y &= y + k(x + x_1) - g(z + z_1), & y_1 &= y_1 - k(x + x_1) + g(z + z_1), \\ z &= z + g(y + y_1) - h(x + x_1), & z_1 &= z_1 - g(y + y_1) + h(x + x_1), \end{aligned}$$

h and k are constants. Show that the line segments joining corresponding points of S and S' are equal and parallel to those for S_1 and S_1 , that the lines joining corresponding points on S and S_1 meet the similar lines for S' and S_1 , and that the common conjugate system on S and S_1 corresponds to the common conjugate system on S' and S_1

Apply the results of Ex 13 to the surfaces of translation

$$\begin{aligned} u^2 - v^2 + 2av, \quad y &= 2u^2 + v^2 - 2av - 2 \int \sqrt{b^2 + 3u^2} du, \quad z = 2bu, \\ + 2v^2 - 2av - 2 \int \sqrt{b^2 + 3u^2} du, \quad y_1 &= -u^2 + v^2 + 2 \int \sqrt{b^2 + 3u^2} du, \\ z_1 &= 2 \int \sqrt{a^2 - 3v^2} dv \end{aligned}$$

that when $g = h = 0, k = -1$, the surface S' is an elliptic paraboloid

15 Show that the equations

$$x = \int \sqrt{1 - \frac{V'^2}{a^2}} dv, \quad y = \int \sqrt{1 - a^2 U'^2} du, \quad z = aU + \frac{V}{a},$$

where the accent indicates differentiation with respect to the argument, define a family of applicable surfaces of translation. Apply the results of Ex 12 to this case.

16 Show that when S and S_1 in Exs 12 and 13 are surfaces of translation, and their generating curves correspond, the same is true of S' and S'_1 .

17 If lines be drawn through points of a Bertrand curve parallel to the binormals of the conjugate curve, their locus is applicable to a surface of revolution.

18 If a real ruled surface is applicable to a surface of revolution, it is applicable to the right helicoid or to a hyperboloid of revolution of one sheet (cf Ex 9, §144).

19 A ruled surface can be deformed in an infinity of ways so that a curve not orthogonal to the generators shall be a line of curvature on the new ruled surface, unless the given curve is a geodesic, in the latter case the deformation is unique and the line of curvature is plane.

20 Let P be any point of a twisted curve C , and M_1, M_2 points on the principal normal to C such that

$$PM_1 = -PM_2 = a \sin \left(\int \frac{ds}{\rho} + b \right),$$

where a, b are constants and ρ is the radius of curvature of C . The loci of the lines through M_1 and M_2 parallel to the tangent to C at P are applicable ruled surfaces.

21 On the surface whose equations are

$$x = u, \quad y = f(u) \phi'(v) + \psi'(v), \quad z = f(u) [\phi(v) - v \phi'(v)] + \psi(v) - v \psi'(v),$$

the parametric curves form a conjugate system, the curves $u = \text{const}$ lie in planes parallel to the yz plane, and the curves $v = \text{const}$ in planes parallel to the x -axis, hence the tangents to the curves $u = \text{const}$ at their points of intersection with a curve $v = \text{const}$ are parallel.

22 Investigate the character of the surfaces of Ex 21 in the following cases (a), $\phi(v) = \sqrt{v^2 + 1}$, (b), $\phi(v) = \text{const}$, (c), $\psi(v) = 0$, (d), $f(u) = au + b$.

23 If the equations of Ex 21 be written

$$x = u, \quad y = f(u) \phi_1(v) + \psi_1(v), \quad z = f(u) \phi_2(v) + \psi(v),$$

the most general applicable surfaces of the same kind with parametric curves corresponding are defined by

$$x_1 = \int \sqrt{1 + \kappa f'^2(u)} du, \quad y_1 = f(u) \Phi_1(v) + \Psi_1(v), \quad z_1 = f(u) \Phi(v) + \Psi(v),$$

where κ is a parameter, and the functions $\Phi_1, \Phi, \Psi_1, \Psi$ satisfy the conditions

$$\Phi_1^2 + \Phi_2^2 = \Phi_1^2 + \Phi_2^2 - \kappa, \quad \Phi_1^2 + \Phi_2^2 = \Phi_1^2 + \Phi_2^2,$$

$$\Psi_2' = \frac{\Phi_1'(\phi_2 \psi_2 + \phi_1 \psi_1') - \Phi_1(\phi \psi_2 + \phi_1 \psi_1')}{\Phi_2 \Phi_1 - \Phi_2 \Phi_1}$$

$$\Psi_1' = \frac{\Phi_2(\phi_2 \psi_2' + \phi_1 \psi_1') - \Phi_2(\phi \psi_2 + \phi_1 \psi_1')}{\Phi_2 \Phi_1 - \Phi_2 \Phi_1}$$

Show also that the determination of Φ_1 and Φ_2 requires only quadrature.

CHAPTER X

DETERMINATION OF SURFACES THE METHOD OF WEINGARTEN

Reduced form of the linear element Weingarten has re-
 that when we reduce the determination of all surfaces appli-
 a given one to the solution of the equation (IX, 18), namely

$$\Delta_{22}\theta = (1 - \Delta_1\theta) K,$$

no use of our knowledge of the given surface, and in
 re trying to solve the problem of finding all the surfaces
 assigned linear element In his celebrated memoir, *Sur la*
ion des surfaces,* which was awarded the grand prize of
 ich Academy in 1894, Weingarten showed that by taking
 of the given surface the above equation can be replaced
 er which can be solved in several important cases This
 is devoted to the exposition of this method We begin by
 ung a particular moving trihedral for the given surface
 lows from (VII, 64) that the necessary and sufficient con-
 at the directrix of a ruled surface be the line of striction is

$$b = x'_0l' + y'_0m' + z'_0n' = 0$$

ctions l', m', n' are proportional to the direction cosines of
 e in which the director-cone of the surface meets the unit
 ith center at the vertex of the cone We call this curve
spherical indicatrix of the surface From (2) and the identity

$$ll' + mm' + nn' = 0$$

n that the tangent to the spherical indicatrix is perpen-
 to the tangent plane to the surface at the corresponding
 the line of striction This fact is going to enable us
 mine under what conditions a ruled surface Σ , tangent
 rved surface S along a curve C , admits the latter for
 of striction

* *Acta Mathematica* Vol XX (1896), pp 159-200

We suppose that the parameters u, v are any whatever, and that the surface is referred to a moving trihedral. We consider the ruled surface formed by the x -axis of the trihedral as the origin of the latter describes the curve C . The point $(1, 0, 0)$ of a second trihedral parallel to this one, but with origin fixed, describes the spherical indicatrix of Σ . From equations (V, 51) we find that the components of a displacement of this point are

$$0, \quad r du + r_1 dv, \quad -(q du + q_1 dv)$$

In order that the displacement be perpendicular to the tangent plane to Σ at the corresponding point of C , that is, perpendicular to the xy -plane of the moving trihedral, we must have

$$(3) \quad r du + r_1 dv = 0$$

Hence if a trihedral T be associated with a surface S in any manner, as the vertex of T describes an integral curve of equation (3), the x -axis of T generates a ruled surface whose line of striction is this curve.

When the parametric lines on S are given, and also the angle U which the x -axis of T makes with the tangent to the curve $v = \text{const}$, the functions r and r_1 are completely determined, as follows from (V, 52, 55). They are

$$(4) \quad r = \frac{H}{E} \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} - \frac{\partial U}{\partial u}, \quad r_1 = \frac{H}{E} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} - \frac{\partial U}{\partial v}$$

Hence if U be given the value

$$(5) \quad U = \int \frac{H}{E} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} dv + \phi(u),$$

where $\phi(u)$ denotes an arbitrary function of u , the function r_1 is zero, and as the vertex of the trihedral describes a curve $u = \text{const}$, the x -axis describes a ruled surface whose line of striction is this curve.

Suppose now that the trihedral is such that $r_1 = 0$. From (V, 48, 64) it follows that

$$(6) \quad \frac{\partial r}{\partial v} = HK,$$

consequently

$$(7) \quad r = \int HK dv + \psi(u),$$

where ψ is an arbitrary function of u .

the right-hand member of (7) be denoted by $f(u, v)$, and change the parameters of the surface in accordance with the equations

$$u_1 = u, \quad v_1 = f(u, v)$$

§ 2 and equation (7) it follows that

$$H = H_1 \frac{\partial f}{\partial v} = HH_1 K$$

is unaltered by the transformation, in terms of the new parameters $H_1 K$ is equal to unity, and hence from (6) we have $H = 1$. Therefore the coordinate curves and the moving trihedral face can be chosen in such a way that

$$r_1 = 0, \quad r = v, \quad HK = 1$$

we may say that the linear element of the surface is in its normal form. It should be remarked that for surfaces of negative curvature the parameters are imaginary.

General formulas If $X_1, Y_1, Z_1, X_2, Y_2, Z_2, X, Y, Z$ denote the direction-cosines of the axes of the moving trihedral with respect to fixed axes, we have, from (V, 47),

$$\begin{cases} \frac{\partial X_1}{\partial u} = X_2 v - X q, & \frac{\partial X_2}{\partial u} = X p - X_1 v, & \frac{\partial X}{\partial u} = X_1 q - X_2 p, \\ \frac{\partial X_1}{\partial v} = -X q_1, & \frac{\partial X_2}{\partial v} = X p_1, & \frac{\partial X}{\partial v} = X_1 q_1 - X_2 p_1 \end{cases}$$

where the functions p, p_1, q, q_1 satisfy equations (V, 48) in the reduced form

$$\frac{\partial p}{\partial v} - \frac{\partial p_1}{\partial u} = -v q_1, \quad \frac{\partial q}{\partial v} - \frac{\partial q_1}{\partial u} = v p_1, \quad p q_1 - p_1 q = 1$$

the coordinates x, y, z of S with reference to these fixed axes are

$$\begin{cases} x = \int (\xi X_1 + \eta X_2) du + (\xi_1 X_1 + \eta_1 X_2) dv, \\ y = \int (\xi Y_1 + \eta Y_2) du + (\xi_1 Y_1 + \eta_1 Y_2) dv, \\ z = \int (\xi Z_1 + \eta Z_2) du + (\xi_1 Z_1 + \eta_1 Z_2) dv, \end{cases}$$

$$\xi^2 + \eta^2 = E, \quad \xi \xi_1 + \eta \eta_1 = F, \quad \xi_1^2 + \eta_1^2 = G,$$

$$\begin{cases} \frac{\partial \xi}{\partial v} - \frac{\partial \xi_1}{\partial u} = -v \eta_1, & \frac{\partial \eta}{\partial v} - \frac{\partial \eta_1}{\partial u} = v \xi_1, \\ p \eta_1 - \eta p_1 + \xi q_1 - q \xi_1 = 0 \end{cases}$$

Weingarten's method consists in replacing the coefficients of ξ, η, ξ_1, η_1 in the last of equations (13) by differential parameters of u formed with respect to the linear element of the spherical representation of the x -axis of the moving trihedral

By means of (9) this linear element is reducible to

$$d\sigma^2 = dX_1^2 + dY_1^2 + dZ_1^2 = (v^2 + q^2) du^2 + 2qq_1 dudv + q_1^2 dv^2$$

the differential parameters of u , formed with respect to this form, have the values *

$$(15) \quad \begin{cases} \Delta_1 u = \frac{1}{v^2}, & \Delta_1 \Delta_1 u = \frac{4(v^2 + q^2)}{v^3 q_1^2}, \\ \Delta_1(u, \Delta_1 u) = \frac{2q}{v^3 q_1}, & \Delta_2 u = \frac{q}{v^3 q_1} - \frac{p_1}{v q_1} \end{cases}$$

Because of the identity (V, 38)

$$\Delta_{22}\theta = \frac{2\Delta_2\theta \Delta_1(\theta, \Delta_1\theta) - \Delta_1\Delta_1\theta}{4\Delta_1\theta},$$

we have also

$$(16) \quad \Delta_{22}u = -\frac{p}{v^4 q_1}$$

If the last of equations (13) be divided by q_1 , and the values of $p/q_1, q/q_1, p_1/q_1$ obtained from (15) and (16) be substituted, we have

$$(17) \quad \eta_1 \Delta_{22}u - \frac{\eta}{v^3} \Delta_2 u - \frac{\xi}{v^4} + \frac{\eta + \xi_1 v^2}{2v} \Delta_1(u, \Delta_1 u) = 0$$

In consequence of the first of equations (15), written

$$(18) \quad v = \frac{1}{\sqrt{\Delta_1 u}},$$

the coefficients of ξ, η, ξ_1, η_1 in (17) are expressible in terms of differential parameters of u formed with respect to (14), as was to be proved

An exceptional case is that in which $q_1 = 0$. Under this condition the spherical representation of the x axis reduces to a curve, as is seen from (14)

* Previously we have indicated by a prime differential parameters formed with respect to the linear element of the spherical representation. For the sake of simplicity we disregard this practice in this chapter

means of (9) we find that

$$\Delta_1(X_1, u) = \frac{X_2}{v}, \quad \Delta_1(Y_1, u) = \frac{Y_2}{v}, \quad \Delta_1(Z_1, u) = \frac{Z_2}{v},$$

sequently equations (11) may be written

$$= \int [\xi X_1 + \eta v \Delta_1(X_1, u)] du + [\xi_1 X_1 + \eta_1 v \Delta_1(X_1, u)] dv,$$

ilarly for y and z

The theorem of Weingarten Equation (17) is the equation Weingarten has suggested as a substitute for equation (1) since that ξ, η, ξ_1, η_1 are known functions of u and v when space S is given. By means of (18) equation (17) can be in a form which involves only u and differential parameters determined with respect to (14). On account of the invariance of these differential parameters this linear element can be expressed in terms of any parameters, say u' and v' . We will show that each solution of equation (17) determines a surface applicable to S . We formulate the theorem of Weingarten as follows

Let S be a surface whose linear element in the reduced form is

$$ds^2 = (\xi^2 + \eta^2) du^2 + 2(\xi\xi_1 + \eta\eta_1) du dv + (\xi_1^2 + \eta_1^2) dv^2,$$

where ξ, η, ξ_1, η_1 are functions of u and v such that

$$\frac{\partial \xi}{\partial v} - \frac{\partial \xi_1}{\partial u} = -\eta\eta_1, \quad \frac{\partial \eta}{\partial v} - \frac{\partial \eta_1}{\partial u} = \xi\xi_1$$

more, let X_1, Y_1, Z_1 be the coordinates of a point on the unit sphere expressed in terms of any two parameters u' and v' , the linear element of the sphere being

$$d\sigma'^2 = \mathcal{C}' du'^2 + 2\mathcal{F}' du' dv' + \mathcal{G}' dv'^2$$

then the integral u_1 of the equation

$$\left(u, \frac{1}{\sqrt{\Delta_1 u}}\right) \Delta_2 u - \eta \left(u, \frac{1}{\sqrt{\Delta_1 u}}\right) (\Delta_1 u)^{\frac{1}{2}} \Delta_2 u - \xi \left(u, \frac{1}{\sqrt{\Delta_1 u}}\right) (\Delta_1 u)^{\frac{1}{2}} + \frac{1}{2} \left[\eta \left(u, \frac{1}{\sqrt{\Delta_1 u}}\right) (\Delta_1 u)^{\frac{1}{2}} + \xi_1 \left(u, \frac{1}{\sqrt{\Delta_1 u}}\right) (\Delta_1 u)^{-\frac{1}{2}} \right] \Delta_1(u, \Delta_1 u) = 0,$$

the differential parameters being formed with respect to (23), renders the following expression and similar ones in y and z total differentials

$$(25) \quad dx = \left[\xi(u_1, v_1) X_1(u', v') + \eta(u_1, v_1) \frac{\Delta_1(X_1, u_1)}{\sqrt{\Delta_1 u_1}} \right] du_1 \\ + \left[\xi_1(u_1, v_1) X_1(u', v') + \eta_1(u_1, v_1) \frac{\Delta_1(X_1, u_1)}{\sqrt{\Delta_1 u_1}} \right] dv_1,$$

where

$$v_1 = \frac{1}{\sqrt{\Delta_1 u_1}},$$

and the surface whose coordinates are the functions x, y, z thus defined has the linear element (21)

Before proving this theorem we remark that the parameters u' and v' may be chosen either as known functions of u and v , or in such a way that the linear element (14) shall have a particular form. In the former case X_1, Y_1, Z_1 are known as functions of u' and v' , and in the second their determination requires the solution of a Riccati equation. However, in what follows we assume that X_1, Y_1, Z_1 are known.

Suppose now that u' and v' are any parameters whatever, and that we have a solution u_1 of equation (24), where the differential parameters are formed with respect to (23). Let v_1 denote the quantity $(\Delta_1 u_1)^{-\frac{1}{2}}$. Both u_1 and v_1 are functions of u' and v' , and consequently the latter are expressible as functions of the former. We express X_1, Y_1, Z_1 as functions of u_1 and v_1 and determine the corresponding linear element of the unit sphere, which we write

$$(26) \quad d\sigma_1^2 = \mathcal{E}_1 du_1^2 + 2\mathcal{F}_1 du_1 dv_1 + \mathcal{G}_1 dv_1^2$$

In terms of u_1 and v_1 we have

$$\frac{1}{v_1^2} = \Delta_1 u_1 = \frac{\mathcal{G}_1}{\mathcal{F}_1^2}, \quad \Delta_1(X_1, u_1) = \frac{\mathcal{G}_1 \frac{\partial X_1}{\partial u_1} - \mathcal{F}_1 \frac{\partial X_1}{\partial v_1}}{\mathcal{F}_1}$$

From these expressions it follows that if we put

$$X_2 = v_1 \Delta_1(X_1, u_1), \quad Y_2 = v_1 \Delta_1(Y_1, u_1), \quad Z_2 = v_1 \Delta_1(Z_1, u_1),$$

we have

$$(27) \quad \begin{cases} \sum X_2^2 = 1, & \sum X_1 X_2 = 0, \\ \sum X_2 \frac{\partial X_1}{\partial u_1} = v_1, & \sum X_2 \frac{\partial X_1}{\partial v_1} = 0 \end{cases}$$

we put

$$= Y_1 Z_2 - Z_1 Y_2, \quad Y = Z_1 X_2 - X_1 Z_2, \quad Z = X_1 Y_2 - Y_1 X_2,$$

tions $X_1, Y_1, \dots, Z$ satisfy a set of equations similar to equations (47)

sequence of (27) the corresponding rotations have the values

$$\begin{aligned} \bar{p} &= - \sum X_2 \frac{\partial X}{\partial u_1}, & \bar{q} &= \sum X_1 \frac{\partial X}{\partial u_1}, \\ \bar{p}_1 &= - \sum X_2 \frac{\partial X}{\partial v_1}, & \bar{q}_1 &= \sum X_1 \frac{\partial X}{\partial v_1}, \\ \bar{r} &= v_1, & \bar{r}_1 &= 0 \end{aligned}$$

it is easily shown that these functions satisfy equations similar

to the functions ξ, η, ξ_1, η_1 are of the same form in (25) as equations similar to the first two of equations (18) are necessitated. Hence the only other equation to be satisfied, in addition to the expressions (25) be exact differentials, is

$$\bar{p}\eta_1(u_1, v_1) - \bar{p}_1\eta(u_1, v_1) + \bar{q}_1\xi(u_1, v_1) - \bar{q}\xi_1(u_1, v_1) = 0$$

it can be shown that the coefficients of (26) are expressible in

$$\mathcal{E}_1 = v_1^2 + \bar{q}^2, \quad \mathcal{F}_1 = \bar{q}\bar{q}_1, \quad \mathcal{G}_1 = \bar{q}_1^2,$$

by means of differential parameters of u_1 formed with reference to (26) the equation (28) can be given the form (17) all the conditions are satisfied, and the theorem of Weingarten has been established.

Other forms of the theorem of Weingarten It is readily seen that equations (22) are satisfied by the expressions

$$\begin{cases} \xi = \frac{\partial' \phi}{\partial u^2} + v^3 \frac{\partial \phi}{\partial v}, & \xi_1 = \frac{\partial^2 \phi}{\partial u \partial v}, \\ \eta = -v \frac{\partial' \phi}{\partial u \partial v}, & \eta_1 = -3v \frac{\partial \phi}{\partial v} - v^2 \frac{\partial^2 \phi}{\partial v^2}, \end{cases}$$

is any function of u and v . Since now

$$\eta + v^3 \xi_1 = 0,$$

equation (17) reduces to

$$3v \frac{\partial \phi}{\partial v} + v^2 \frac{\partial^2 \phi}{\partial v^2} \Delta_{22} u - \frac{1}{v} \frac{\partial' \phi}{\partial u \partial v} \Delta_2 u + \frac{1}{v^4} \left(\frac{\partial^2 \phi}{\partial u^2} + v^3 \frac{\partial \phi}{\partial v} \right) = 0$$

This equation will be simplified still more by the introduction of two new parameters which are suggested by the following considerations

As previously defined, the functions X_1, Y_1, Z_1 are the direction-cosines of lines tangent to the given surface S in such a way that the ruled surface formed by these tangents at points of a curve $u = \text{const}$ has this curve for its line of striction. Moreover, from the theorem of Weingarten it follows that the functions X_1, Y_1, Z_1 have the same significance for the surface applicable to S which corresponds to a particular solution of equation (17)

X_1, Y_1, Z_1 may be taken also as the direction cosines of the normal to a large group of surfaces, as shown in § 67. In particular, consider the surface Σ which is the envelope of the plane

$$X_1x + Y_1y + Z_1z = u$$

Each solution of equation (17) determines such a surface. If $\bar{x}, \bar{y}, \bar{z}$ denote the coordinates of the point of contact of this plane with Σ , we have from (V, 32)

$$(32) \quad \bar{x} = uX_1 + \Delta_1(u, X_1),$$

which, in consequence of (19), may be written

$$(32') \quad \bar{x} = uX_1 + \frac{1}{v}X$$

Hence the point of contact of Σ lies in the plane through the origin parallel to the tangent plane to S at the corresponding point.

If the square of the distance of the point of contact from the origin be denoted by $2q$, and the distance from the origin to the tangent plane by p ,* we have

$$(33) \quad 2q = \bar{x}^2 + \bar{y}^2 + \bar{z}^2 = u^2 + \frac{1}{v^2}, \quad p = u$$

From (V, 35, 37) it follows that the principal radii of Σ are given by

$$(34) \quad \begin{cases} \rho_1 + \rho_2 = -(\Delta_2 u + 2u), \\ \rho_1 \rho_2 = \Delta_2 u + u \Delta_1 u + u^2, \end{cases}$$

* The reader will observe that the functions p and q thus defined are different from the notations designated by the same letters. As this notation is generally employed in the treatment of the theorem of Weingarten it has seemed best to retain it, even at the risk of a confusion of notation.

the differential parameters are formed with respect to (14) these equations we have

$$\begin{cases} \Delta_2 u = -(\rho_1 + \rho_2) - 2u, \\ \Delta_{22} u = \rho_1 \rho_2 + (\rho_1 + \rho_2)u + u^2 \end{cases}$$

we shall now effect a change of parameters, using p and q as the new ones. By direct calculation we obtain

$$\begin{cases} \frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial p} + \frac{\partial \phi}{\partial q} p, & \frac{\partial \phi}{\partial v} = -\frac{1}{v^3} \frac{\partial \phi}{\partial q}, \\ \frac{\partial^2 \phi}{\partial u^2} = \frac{\partial^2 \phi}{\partial p^2} + 2p \frac{\partial^2 \phi}{\partial p \partial q} + p^2 \frac{\partial^2 \phi}{\partial q^2} + \frac{\partial \phi}{\partial q}, \\ \frac{\partial^2 \phi}{\partial u \partial v} = -\frac{1}{v^3} \left(\frac{\partial^2 \phi}{\partial p \partial q} + \frac{\partial^2 \phi}{\partial q^2} p \right), \\ \frac{\partial^2 \phi}{\partial v^2} = \frac{1}{v^5} \frac{\partial^2 \phi}{\partial q^2} + \frac{3}{v^4} \frac{\partial \phi}{\partial q} \end{cases}$$

means of the equations (33) and (36) the fundamental equation (31) can be reduced to

$$\frac{\partial^2 \phi}{\partial q^2} \rho_1 \rho_2 - \frac{\partial^2 \phi}{\partial p \partial q} (\rho_1 + \rho_2) + \frac{\partial^2 \phi}{\partial p^2} = 0$$

is the form in which the fundamental equation was first considered by Weingarten*. The method of §§ 146, 147 was a subdevelopment

in terms of the parameters p and q the formulas (29) become

$$\begin{cases} \xi = \frac{\partial^2 \phi}{\partial p^2} + 2p \frac{\partial^2 \phi}{\partial p \partial q} + p^2 \frac{\partial^2 \phi}{\partial q^2}, & \eta = \sqrt{2q - p^2} \left(\frac{\partial^2 \phi}{\partial p \partial q} + p \frac{\partial^2 \phi}{\partial q^2} \right), \\ \xi_1 = -(2q - p^2)^{\frac{3}{2}} \left(\frac{\partial^2 \phi}{\partial p \partial q} + p \frac{\partial^2 \phi}{\partial q^2} \right), & \eta_1 = -(2q - p^2)^{\frac{1}{2}} \frac{\partial^2 \phi}{\partial q^2} \end{cases}$$

these values and the expression for $\Delta_1(u, \lambda_1)$ given by (32) be substituted in (20), it is reducible to

$$x = \int \left(X_1 \frac{\partial^2 \phi}{\partial p^2} + \bar{x} \frac{\partial^2 \phi}{\partial p \partial q} \right) dp + \left(X_1 \frac{\partial^2 \phi}{\partial p \partial q} + \bar{x} \frac{\partial^2 \phi}{\partial q^2} \right) dq$$

Hence the equations for S may be written

$$(39) \quad \begin{cases} dx = X_1 d\left(\frac{\partial\phi}{\partial p}\right) + \bar{x} d\left(\frac{\partial\phi}{\partial q}\right), \\ dy = Y_1 d\left(\frac{\partial\phi}{\partial p}\right) + \bar{y} d\left(\frac{\partial\phi}{\partial q}\right), \\ dz = Z_1 d\left(\frac{\partial\phi}{\partial p}\right) + \bar{z} d\left(\frac{\partial\phi}{\partial q}\right), \end{cases}$$

and consequently the linear element of S is of the form

$$(40) \quad ds^2 = \left[d\left(\frac{\partial\phi}{\partial p}\right) \right]^2 + 2p d\left(\frac{\partial\phi}{\partial p}\right) d\left(\frac{\partial\phi}{\partial q}\right) + 2q \left[d\left(\frac{\partial\phi}{\partial q}\right) \right]^2$$

Since these various expressions and equations differ only in form from those which figure in the theorem of Weingarten, the latter is just as true for these new equations. We remark also that the right-hand member of (40) depends only upon the form of ϕ . Hence we have the theorem of Weingarten in the form

When ϕ in equation (37) is a definite function of p and q , this equation defines a large group of surfaces with the same spherical representation, the functions ρ_1 and ρ denoting the principal radii and p and $2q$ the distance from the origin to the tangent plane and the square of the distance to the point of contact. Each surface Σ satisfying this condition gives by quadratures (39) a surface with the linear element (40). Conversely, each surface with this linear element stands in such relation to some surface satisfying the corresponding equation (37).

As a corollary to the preceding results, we have the theorem

The linear element of any surface S is reducible to the form

$$(41) \quad ds^2 = du^2 + 2 \int_{cu}^c \psi du dv + 2 \int_{cu}^c \psi dv^2,$$

where ψ is a function of u and v .

For, we have seen that the linear element of any surface is reducible to the form (40). If, then, we change the parameters by means of the equations

$$(42) \quad u = \frac{c\phi}{cp}, \quad v = \frac{c\phi}{cq},$$

we have

$$(43) \quad ds^2 = du^2 + 2p du dv + 2q dv^2$$

(42) it follows that

$$du = \frac{\partial^2 \phi}{\partial p^2} dp + \frac{\partial^2 \phi}{\partial p \partial q} dq, \quad dv = \frac{\partial^2 \phi}{\partial p \partial q} dp + \frac{\partial^2 \phi}{\partial q^2} dq,$$

sequently

$$\begin{cases} \frac{\partial p}{\partial u} = \frac{\partial^2 \phi}{\partial q^2} \Delta, & \frac{\partial p}{\partial v} = -\frac{\partial^2 \phi}{\partial p \partial q} \Delta, \\ \frac{\partial q}{\partial u} = -\frac{\partial^2 \phi}{\partial p \partial q} \Delta, & \frac{\partial q}{\partial v} = \frac{\partial^2 \phi}{\partial p^2} \Delta, \end{cases}$$

$$\frac{1}{\Delta} = \frac{\partial^2 \phi}{\partial p^2} \frac{\partial^2 \phi}{\partial q^2} - \left(\frac{\partial^2 \phi}{\partial p \partial q} \right)^2$$

(44) it is seen that $\frac{\partial p}{\partial v} = \frac{\partial q}{\partial u}$, and consequently the inverse relations (42) are of the form

$$p = \frac{\partial \psi}{\partial u}, \quad q = \frac{\partial \psi}{\partial v}$$

equation (43) is of the form (41), as was to be proved
er, equations (44) reduce to

$$\frac{\partial^2 \psi}{\partial u^2} = \frac{\partial^2 \phi}{\partial q^2} \Delta, \quad \frac{\partial^2 \psi}{\partial u \partial v} = -\frac{\partial^2 \phi}{\partial p \partial q} \Delta, \quad \frac{\partial^2 \psi}{\partial v^2} = \frac{\partial^2 \phi}{\partial p^2} \Delta$$

rms of these parameters u, v equations (39) reduce to

$$x = X_1 du + \bar{x} dv, \quad dy = Y_1 du + \bar{y} dv, \quad dz = Z_1 du + \bar{z} dv$$

the coordinates of Σ are given by

$$\bar{x} = \frac{\partial x}{\partial v}, \quad \bar{y} = \frac{\partial y}{\partial v}, \quad \bar{z} = \frac{\partial z}{\partial v},$$

direction-cosines of the normal to Σ are

$$X_1 = \frac{\partial x}{\partial u}, \quad Y_1 = \frac{\partial y}{\partial u}, \quad Z_1 = \frac{\partial z}{\partial u},$$

the normals to Σ are parallel to the corresponding tangents
urves $v = \text{const}$ on S . Hence we have the following theorem

the linear element of a surface is in the form (41), the surface whose coordinates are given by (48) has the same spherical

DEFORMATION OF SURFACES

of its normals as the tangents to the curves $v = \text{const}$ and $2q$ denote the distance from the origin to the tangent and the square of the distance to the point of contact, they are (45) Moreover, if the change of parameters defined above be expressed in the inverse form

$$u = \frac{\partial \phi}{\partial p}, \quad v = \frac{\partial \phi}{\partial q},$$

the radii of Σ satisfy the condition

$$\rho_1 \rho_2 \frac{\partial^2 \phi}{\partial q^2} - (\rho_1 + \rho_2) \frac{\partial^2 \phi}{\partial p \partial q} + \frac{\partial^2 \phi}{\partial p^2} = 0,$$

the coordinates of S are given by quadratures of the form

$$(52) \quad dx = X_1 d\left(\frac{\partial \phi}{\partial p}\right) + \bar{x} d\left(\frac{\partial \phi}{\partial q}\right)$$

Moreover, every surface with the same representation as Σ , and whose functions ρ_1, ρ_2, p, q satisfy (51) for the same ϕ , determines by equations of the form (52) a surface applicable to S^*

149 Surfaces applicable to a surface of revolution When the linear element of a surface applicable to a surface of revolution is written

$$(53) \quad ds^2 = du^2 + \rho^2(u_1) dv_1^2,$$

and the x -axis of the moving trihedral is tangent to the curve $v = \text{const}$, the function r is equal to zero, as follows from (4)

In order to obtain the conditions (8), we effect the transformation of variables

$$\bar{u} = v_1, \quad \bar{v} = -u_1,$$

so that the linear element becomes

$$(54) \quad ds^2 = \rho^2 du^2 + dv^2$$

Now $r = \rho'$, $r_1 = 0$, and consequently in order to have the linear element in the reduced form we must take

$$(55) \quad u = \bar{u}, \quad v = \rho'(-v)$$

* For a direct proof of this theorem the reader is referred to a memoir by Goursat, Sur un théorème de M. Weingarten et sur la théorie des surfaces applicables, *Toulouse Annales*, Vol. V (1891), also Darboux, Vol. IV, p. 316, and Bianchi, Vol. II, p. 198

From these results and (32') we find that the coordinates of the Σ are given by

$$\bar{x} = -\frac{1}{\rho'} \frac{\partial x}{\partial u_1} + \frac{v_1}{\rho} \frac{\partial x}{\partial v_1}, \quad \bar{y} = -\frac{1}{\rho'} \frac{\partial y}{\partial u_1} + \frac{v_1}{\rho} \frac{\partial y}{\partial v_1},$$

$$\bar{z} = -\frac{1}{\rho'} \frac{\partial z}{\partial u_1} + \frac{v_1}{\rho} \frac{\partial z}{\partial v_1},$$

the direction-cosines of the normals to Σ are

$$X_1 = \frac{1}{\rho} \frac{\partial x}{\partial v_1}, \quad Y_1 = \frac{1}{\rho} \frac{\partial y}{\partial v_1}, \quad Z_1 = \frac{1}{\rho} \frac{\partial z}{\partial v_1}$$

we have

$$p = \sum \bar{x} X_1 = v_1, \quad 2q = \sum \bar{x}^2 = v_1^2 + \frac{1}{\rho'^2}$$

we have the theorem

curve which is the deform of a meridian of a surface of revolution corresponds on the surface Σ a curve such that the tangent to Σ at points of the curve are at a constant distance from the and to a deform of a parallel there corresponds a curve such that the projection of the radius vector upon the tangent plane at a point is constant

In the present case $\eta = \xi_1 = 0$, consequently we have, from (38),

$$\frac{\partial^2 \phi}{\partial p \partial q} + p \frac{\partial^2 \phi}{\partial q^2} = 0$$

equation is satisfied by

$$\phi(p, q) = f(2q - p^2),$$

f is any function whatever. In terms of this function we have from (38),

$$\xi = -2f', \quad \eta_1 = -4(2q - p^2)^2 f'',$$

the accents indicate differentiation with respect to the argument, $2q - p^2$

By means of (55) the linear element (54) can be transformed into

$$ds^2 = \omega^2(v) du^2 + \frac{\omega'^2(v)}{v^2} dv^2,$$

where $\omega(v)$ being defined by

$$\omega(v) = \rho(-\bar{v})$$

Since $\eta = \xi_1 = 0$, we have

$$\xi = \omega(v), \quad \eta_1 = -\frac{\omega'(v)}{v},$$

and we know that $r = v$. Now equations (58) become

$$\omega(v) = -2f', \quad \omega'(v) = 4v(2q - p^2)^2 f'',$$

and these are consistent because of the relation $2q - p^2 = 1/v$ which results from (56). Hence we have the theorem

When $\phi(p, q)$ is a function of $2q - p^2$, the corresponding surface is applicable to a surface of revolution, the tangents to the deforms of the parallels being parallel to the corresponding normals to Σ .

If we give ϕ the form (57) and put $\psi = 2f'$, the linear element of S is

$$(59) \quad ds^2 = (2q - p^2) d\psi^2 + \psi^2 dp^2,$$

as follows from (40) or (58).

150 Minimal lines on the sphere parametric In § 147 we remarked that the parametric curves on the sphere may be any whatever. An interesting case is that in which they are the imaginary generatrices. In § 35 we saw that the parameters of these lines, say α and β , can be so chosen that

$$(60) \quad X_1 = \frac{\alpha + \beta}{1 + \alpha\beta}, \quad Y_1 = \frac{(\beta - \alpha)}{1 + \alpha\beta}, \quad Z_1 = \frac{\alpha\beta - 1}{1 + \alpha\beta}$$

Consequently

$$(61) \quad d\sigma^2 = d\lambda_1^2 + d\lambda_1'^2 + d\lambda_1''^2 = \frac{4 d\alpha d\beta}{(1 + \alpha\beta)^2}$$

From (32) we find that the coordinates of Σ , the envelope of the plane

$$\lambda_1 x + \lambda_1 y + \lambda_1 z - p = 0,$$

are

$$(62) \quad \begin{cases} \bar{x} = p\lambda_1 + \frac{1}{2} \left[(1 - \alpha) \frac{\rho'}{\alpha} + (1 - \beta) \frac{\rho'}{\beta} \right], \\ y = p\lambda_1 + \frac{1}{2} \left[(1 + \alpha) \frac{\rho'}{\alpha} - (1 + \beta) \frac{\rho'}{\beta} \right], \\ z = p\lambda_1 + \alpha \frac{\rho'}{\alpha} + \beta \frac{\rho'}{\beta} \end{cases}$$

From these we obtain

$$(63) \quad 2q = p^2 + (1 + \alpha\beta)^2 \frac{\rho'}{\partial\alpha} \frac{\rho'}{\partial\beta}$$

means of (34) the expressions for $\rho_1 + \rho_2$ and $\rho_1 \rho_2$ in terms of its derivatives with respect to α and β can be readily found and thus the fundamental equation (37) put in a new form. In this case, it is not with the general case that we shall now concern ourselves, but with a particular form of the function $\phi(p, q)$. This function has been considered by Weingarten*, it is

$$\phi(p, q) = pq - \frac{p^3}{3} - \bar{\omega}(p)$$

In this case

$$\frac{\partial^2 \phi}{\partial p^2} = -2p - \bar{\omega}''(p), \quad \frac{\partial^2 \phi}{\partial p \partial q} = 1, \quad \frac{\partial^2 \phi}{\partial q^2} = 0,$$

$$\Delta_2 p = (1 + \alpha\beta)^2 \frac{\partial^2 p}{\partial \alpha \partial \beta},$$

Equation (37) reduces to

$$\rho_1 + \rho_2 = -(2p + \bar{\omega}''(p)),$$

In consequence of (34), may be written

$$\frac{\partial^2 p}{\partial \alpha \partial \beta} = \frac{\bar{\omega}''(p)}{(1 + \alpha\beta)^2}$$

If the values from (62) are substituted in (52), we obtain

$$\begin{cases} x = u_1 X_1 - \int u_1 dX_1 + \frac{1}{2} \left[(1 - \alpha^2) \frac{\partial p}{\partial \alpha} + (1 - \beta^2) \frac{\partial p}{\partial \beta} \right] dp, \\ y = u_1 Y_1 - \int u_1 dY_1 + \frac{1}{2} \left[(1 + \alpha^2) \frac{\partial p}{\partial \alpha} - (1 + \beta^2) \frac{\partial p}{\partial \beta} \right] dp, \\ z = u_1 Z_1 - \int u_1 dZ_1 + \left(\alpha \frac{\partial p}{\partial \alpha} + \beta \frac{\partial p}{\partial \beta} \right) dp, \end{cases}$$

$$u_1 = q - \frac{p^2}{2} - \bar{\omega}'(p) = \frac{(1 + \alpha\beta)^2}{2} \frac{\partial p}{\partial \alpha} \frac{\partial p}{\partial \beta} - \bar{\omega}'(p)$$

(42) and (64) we have

$$u = q - p^2 - \bar{\omega}'(p), \quad v = p$$

The linear element (43) of S is, in this case,

$$ds^2 = du^2 + 2v du dv + 2[u + v^2 + \bar{\omega}'(v)] dv^2$$

* *Acta Mathematica*, Vol XX (1896), p 195

However, from (68) it is seen that

$$(70) \quad u_1 = u + \frac{v^2}{2},$$

so that (69) may be written

$$(71) \quad ds^2 = du_1^2 + 2[u_1 + \bar{\omega}'(v)] dv^2$$

Gathering together these results, we have the theorem

The determination of all the surfaces with the linear element (71) reduces to the integration of the equation

$$(72) \quad \frac{\partial^2 p}{\partial \alpha \partial \beta} = \frac{\bar{\omega}''(p)}{(1 + \alpha\beta)^2}$$

The integral of this equation for $\omega(p)$ arbitrary is not known. However, the integral is known in certain cases. We consider several of these.

151 Surfaces of Goursat Surfaces applicable to certain paraboids. When we take

$$(73) \quad \bar{\omega}'(p) = \frac{1}{2} m(1 - m)p^2,$$

m being any constant, equation (72) becomes

$$(74) \quad \frac{\partial^2 p}{\partial \alpha \partial \beta} = \frac{m(1 - m)p}{(1 + \alpha\beta)^2}$$

The general integral of this equation can be found by the method of Laplace,* in finite form or in terms of definite integrals, according as m is integral or not.

The linear element of the surface S is

$$(75) \quad ds^2 = du_1^2 + [2u_1 + m(1 - m)v] dv^2$$

And the surfaces Σ are such that

$$(76) \quad \rho_1 + \rho_2 + 2p = m(m - 1)p,$$

that is, the sum of the principal radii is proportional to the distance of the tangent plane from a fixed point. These surfaces were first studied by Goursat,† and are called, consequently, *the surfaces of Goursat*.

* Darboux, Vol II, p. 56

† *American Journal* Vol X (1888), p. 187

ix has remarked * that equation (71) is similar to the linear of ruled surfaces (VII, 53) In fact, if the equations of a face are written in the form

$$x = x_0 + lu_1, \quad y = y_0 + mu_1, \quad z = z_0 + nu_1,$$

, l, m, n are functions of v alone, which now is not ly the arc of the directrix, the linear element of the will have the form (71), provided that

$$l'^2 = 1, \quad \Sigma x_0' l = 0, \quad \Sigma x_0' l'^2 = 2 \bar{\omega}'(v), \quad \Sigma x_0' l' = 1, \quad \Sigma l'^2 = 0$$

sequence of the equations

$$\Sigma ll' = 0, \quad \Sigma l'^2 = 0,$$

s that a ruled surface of this kind admits an isotropic vector If this plane be $x + iy = 0$, that is, if

$$l' - m' - n' = 1 \text{ et } 0,$$

$$l = V, \quad m = iV, \quad n = 1,$$

is a function of v By means of these values and equation (77), we can put (77) in the form

$$\begin{cases} x + iy = \int \frac{dv}{V'}, \\ x - iy = 2Vu_1 + 2 \int V' \bar{\omega}' dv - \int \frac{V'^2}{V'} dv, \\ z = u_1 - \int \frac{V}{V'} dv \end{cases}$$

all find that among these surfaces there is an imaginary one to which are applicable certain surfaces to which Weierstrass called attention To this end we consider the function

$$\bar{\omega}'(p) = -\sqrt{\kappa}p - 2\kappa e^{-\frac{2p}{\sqrt{\kappa}}},$$

denotes a constant Now equation (66) becomes

$$\frac{\partial^2 p}{\partial \alpha \partial \beta} = \sqrt{\kappa} \frac{4e^{-\frac{2p}{\sqrt{\kappa}}} - 1}{(1 + \alpha\beta)^2}$$

In consequence of the identity

$$\frac{\partial^2}{\partial \alpha \partial \beta} \log (1 + \alpha \beta)^2 = \frac{2}{(1 + \alpha \beta)^2},$$

the preceding equation is equivalent to

$$\frac{\partial^2}{\partial \alpha \partial \beta} \log (1 + \alpha \beta)^2 e^{\frac{2p}{\sqrt{\kappa}}} = \frac{8}{e^{\frac{2p}{\sqrt{\kappa}}} (1 + \alpha \beta)^2}$$

If we put

$$e^\theta = e^{\frac{p}{\sqrt{\kappa}}} (1 + \alpha \beta),$$

this equation takes the Liouville form

$$\frac{\partial^2 \theta}{\partial \alpha \partial \beta} = 4 e^{-2\theta},$$

of which the general integral is

$$e^{-2\theta} = \frac{1}{4} \frac{A'B'}{(1 + AB)^2},$$

where A and B are functions of α and β respectively, and the accents indicate differentiation with respect to these. Hence the general integral of (81) is

$$e^{\frac{p}{\sqrt{\kappa}}} = \frac{2(1 + AB)}{\sqrt{A'B'}(1 + \alpha \beta)},$$

and the linear element of S is

$$(82) \quad ds^2 = du_1^2 + 2 \left(u_1 - v\sqrt{\kappa} - 2\kappa e^{-\frac{2v}{\sqrt{\kappa}}} \right) dv^2$$

If now, in addition to (80), we take

$$V = \frac{1}{2} e^{\frac{1}{\sqrt{\kappa}}}, \quad \int \frac{V}{V'} dv = v\sqrt{\kappa} - \frac{\kappa}{2},$$

the equations (79) take such a form that

$$(83) \quad (x + iy)x = -\kappa z$$

Hence the surfaces with the linear element (82) are applicable to the imaginary paraboloid (83). The generator $x + iy = 0$ of the paraboloid in the plane at infinity is tangent to the imaginary circle at the point $(x, y, z) = (1, 0)$, which is a different point from that in which the plane at infinity touches the surface; that is, the point of intersection of the two generators

The most interesting case is afforded when m in (73) has the value $\frac{1}{2}$. Then $\bar{\omega}'(v) = -v^2$, and equation (71) becomes

$$ds^2 = du_1^2 + 2(u_1 - v^2)dv^2$$

take $V = v/\sqrt{2\kappa}$, we obtain from equations (79)

$$iy = \sqrt{2\kappa}v, \quad x - iy = \sqrt{\frac{2}{\kappa}}v\left(u_1 - \frac{v^2}{2}\right), \quad z = u_1 - \frac{v^2}{2},$$

which we find, by the elimination of u_1 and v ,

$$(x + iy)z = \kappa(x - iy)$$

The generator $x + iy = 0$ in the plane at infinity on the paraboloid is tangent to the imaginary circle at the point $(x, y, z) = (1, i, 0)$, in the case of the paraboloid (83), but the paraboloid (85) is tangent to the plane at infinity at the same point.

GENERAL EXAMPLES

A moving trihedral can be associated with a surface in an infinity of ways so that the vertex of the trihedral describes a curve $u = \text{const}$ the x axis generates a surface whose line of striction is this curve.

The tangents to the curves $v = \text{const}$ on a surface at the points where these curves are met by an integral curve of the equation

$$\begin{Bmatrix} 11 \\ 2 \end{Bmatrix} du + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} dv = 0$$

is the surface for which the latter curve is the line of striction.

The ruled surface formed by an infinity of tangents to a surface S has the same points of contact for its line of striction, this relation is unaltered by deformations of S .

Now that if D, D', D'' are the second fundamental coefficients of a surface the lineal element (58), the equation of the lines of curvature of the surface Σ is reducible to

$$\begin{vmatrix} D du_1 + D' dv_1 & D du_1 + D' dv_1 \\ \frac{\rho}{\rho'} du_1 & \rho \rho' dv_1 \end{vmatrix} = 0$$

Now that the surface Σ associated by the method of Weingarten with a surface applicable to a surface of revolution corresponds with parallelism of tangent to the surface S complementary to S with respect to the deformations of the surface, and that the lines of curvature on Σ and S correspond.

6 Show that when ϕ has the form (57), the equation (51) is reducible to

$$(\rho_1 + p)(\rho_2 + p) = F(2q - p^2),$$

hence the determination of all the surfaces applicable to surfaces of revolution is equivalent to the determination of those surfaces Σ which are such that if M_1 and M_2 are the centers of principal curvature of Σ at a point M , and N is the projection of the origin O on the normal at M , the product $NM_1 \cdot NM_2$ is a function of ON .

* Given any surface S applicable to a surface of revolution. Draw through point O segments parallel to the tangents to the deformations of the meridian of lengths proportional to the radii of the corresponding parallels, and through the extremities of these segments draw lines parallel to the normals to S . Show that these lines form a normal congruence whose orthogonal surfaces Σ have the same spherical representation of their lines of curvature as S and are integral surfaces of the equation of Ex. 6.

8 Let S be a surface applicable to a surface of revolution and S' the surface complementary to S with respect to the deformations of the meridians, let also Σ and Σ' be surfaces associated with S and S' respectively after the manner of Ex. 7. Show that corresponding normals to Σ and Σ' are perpendicular to one another and that the common perpendicular to these normals passes through the origin and is divided by it into two segments which are functions of one another.

9 Show that a surface determined by the equation

$$2q + \kappa + (\rho_1 + \rho_2)p + \rho_1\rho_2 = 0,$$

where κ is a constant, possesses the property that the sphere described on the segment of each normal between the centers of principal curvature with this segment for diameter cuts the sphere with center at the origin and of radius $\sqrt{\pm \kappa}$ in great circles, orthogonally, or passes through the origin, according as κ is positive, negative, or zero. These surfaces are called the *surfaces of Bianchi*.

10 Show that for the surfaces of Bianchi the function $\phi(p, q)$ is of the form

$$\phi = \sqrt{2q - p^2 + \kappa},$$

and that the linear element of the associated surface S' applicable to a surface of revolution is

$$ds^2 = \left(\frac{1}{\psi^2} - \kappa \right) d\psi^2 + \psi^2 dp^2$$

Show also that according as $\kappa = 0, > 0, < 0$ the linear element of S' is reducible to the respective forms

$$ds^2 = du^2 + e^{2u} dv^2, \quad ds^2 = \tanh^4 u du^2 + \cosh^2 u dv^2, \quad ds^2 = \coth^4 u du^2 + \cosh^2 u dv^2$$

On account of this result and Ex. 10, p. 318 the surfaces of Bianchi are said to be of the *parabolic, elliptic, or hyperbolic type*, according as $\kappa = 0, > 0, < 0$.

11 Let S be a pseudospherical surface with its linear element in the form (VIII, 32), and S_1 the Bianchi transform whose linear element is (VIII, 33). Find the coordinates x, y, z of the surface Σ associated with S_1 by the method of Weingarten, and show that by means of Ex. 8, p. 291, the expression for x is reducible to

$$\bar{x} = ae^{\frac{\xi}{2}} (\cos \theta X_1 + \sin \theta X_2) + \zeta \bar{1},$$

where X_1, X_2, X are the direction-cosines with respect to the x -axis of the tangents to the lines of curvature of S and of the normal to the latter.

that the surfaces S and Σ of Ex 11 have the same spherical representation of their lines of curvature, that Σ is a surface of Bianchi of the parabolic type and consequently there is an infinity of these surfaces of the parabolic type which have the same spherical representation of their lines of curvature as a pseudospherical surface S .

Show that if Σ_1 and Σ_2 are two surfaces of Bianchi of the parabolic type having the same spherical representation of their lines of curvature, the locus which divides in constant ratio the line joining corresponding points of S and a surface of Bianchi with the same representation of its lines of curvature is of the elliptic or hyperbolic type according as the point divides the line internally or externally.

Show that if S is a pseudospherical surface with its linear element in the form $ds^2 = d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2$, the coordinates $\bar{x}_1, \bar{y}_1, \bar{z}_1$ of the surface Σ determined by the method of Ex 11 are reducible to

$$\bar{x}_1 = (ae^{-\frac{\xi}{a}} \cos \theta + \eta \sin \theta) X_1 + (ae^{-\frac{\xi}{a}} \sin \theta - \eta \cos \theta) X_2,$$

where $X_1, Y_1, Z_1, X_2, Y_2, Z_2$ are the direction cosines of the tangents to the lines of curvature of S . Show also that Σ has the same spherical representation of its lines of curvature as the surface S_1 with linear element (VIII, 38)

derived from the equations

$$\bar{x}X_1 + \bar{y}Y_1 + \bar{z}Z_1 = p, \quad \bar{x}^2 + \bar{y}^2 + \bar{z}^2 = 2q,$$

and (44), (48), and (49), the equations

$$\bar{x} \frac{\partial^2 \bar{x}}{\partial u^2} = \Delta \frac{\partial^2 \phi}{\partial q^2}, \quad \sum \bar{x} \frac{\partial^2 \bar{x}}{\partial u \partial v} = -\Delta \frac{\partial^2 \phi}{\partial p \partial q}, \quad \sum \bar{x} \frac{\partial^2 \bar{x}}{\partial v^2} = \Delta \frac{\partial^2 \phi}{\partial p^2}$$

where u, v are the coordinates of S .

Show that the equations for Σ similar to (IV, 27) are reducible to

$$\frac{\partial^2 \bar{x}}{\partial u \partial v} du + \frac{\partial^2 \bar{x}}{\partial v^2} dv + \bar{r} \left(\frac{\partial^2 \bar{x}}{\partial u^2} du + \frac{\partial^2 \bar{x}}{\partial u \partial v} dv \right) = 0,$$

expressions in u and v . Derive therefrom (cf Ex 15) the equations

$$D du + D dv + \bar{r} (D du + D dv) = 0,$$

$$\frac{\partial^2 \phi}{\partial p \partial q} du - \frac{\partial \phi}{\partial p^2} dv - \bar{r} \left(\frac{\partial^2 \phi}{\partial q^2} du - \frac{\partial^2 \phi}{\partial p \partial q} dv \right) = 0,$$

where D, D are the second fundamental coefficients of S .

Show that the lines of curvature on Σ correspond to a conjugate system on S .

Show that for the surface Σ we have

$$\frac{\partial \bar{x}}{\partial p} = \rho_1 \frac{\partial X_1}{\partial q}, \quad \frac{\partial \bar{x}}{\partial q} = \frac{\partial Y_1}{\partial p} - (\rho_1 + \rho_2) \frac{\partial X_1}{\partial q}$$

where S is the surface defined by (67) and S_1 the surface whose coordinates are

$$x_1 = x - u_1 Y_1, \quad y_1 = y - u_1 Y_1, \quad z_1 = z - u_1 Z_1$$

where S_1 is an involute of S , that the curves $p = \text{const}$ are geodesics on S and $q = \text{const}$ are geodesics on S_1 , and that the radii of principal curvature of S_1 are

$$\rho_1 = u_1, \quad \rho_2 = -[u_1 + 2\bar{\omega}(p)]$$

20 Show that when m in (73) is 0 or 1, the function p is the sum of two arbitrary functions of α and β respectively, that the linear element of S is

$$ds^2 = du_1^2 + 2u_1 dv^2,$$

that S is an evolute of a minimal surface (cf Ex 19), and that the mean evolute Σ is a point

21 Show that when m in (73) is 2, the general integral of equation (74) is

$$p = f_1'(\alpha) + f_2'(\beta) - 2 \frac{\beta f_1'(\alpha) + \alpha f_2'(\beta)}{1 + \alpha\beta},$$

where f_1 and f_2 are arbitrary functions of α and β respectively. Show also that the surface Σ is minimal (cf § 161)

22 Show that the mean evolute of a surface of Goursat is a surface of Goursat homothetic to the given one

23 Show that when $\bar{w}(p) = \frac{1}{2}ap^2$, then

$$p = a \log(1 + \alpha\beta) + f_1(\alpha) + f_2(\beta),$$

where f_1 and f_2 are arbitrary functions, that the linear element of S is

$$ds^2 = du_1^2 + 2(u_1 + av) dv^2,$$

and that the mean evolute of Σ is a sphere

24 Show that the surfaces S of Ex 23 are applicable to the surfaces of revolution S_0 whose equations are

$$x_0 = \alpha \bar{u} \cos \frac{\bar{v}}{\alpha}, \quad y_0 = \alpha \bar{u} \sin \frac{\bar{v}}{\alpha}, \quad z_0 = \int \sqrt{u^2 - \alpha^2 - \alpha^2 d\bar{u}},$$

where α is an arbitrary constant. Show also that when $\alpha = u$, S_0 is a paraboloid

25 Show that when

$$\phi(p, q) = \left(q + \frac{m}{2}\right)^2 - mp^2,$$

the surfaces Σ are spherical or pseudospherical according as m is positive or negative, also that the surfaces S are applicable to the surface

$$x + y = v, \quad x - y = \frac{v^2}{2} - \frac{u^2}{2m} - mv, \quad z = u,$$

which is a paraboloid tangent to the plane at infinity at a point of the circle at infinity

CHAPTER XI

INFINITESIMAL DEFORMATION OF SURFACES

General problem The preceding chapters deal with pairs of surfaces which are such that in order that one may be transformed into the other a finite deformation is necessary. In the present chapter we shall be concerned with the infinitesimal deformations which constitute the intermediate steps in such a transformation.

Let x, y, z, x', y', z' respectively be the coordinates of a surface S and S' , the latter being obtained from the former by a small deformation. If we put

$$x' = x + \epsilon x_1, \quad y' = y + \epsilon y_1, \quad z' = z + \epsilon z_1,$$

where ϵ denotes a small constant and x_1, y_1, z_1 are determined functions of x, y, z , these functions are proportional to the direction-cosines of the line through corresponding points of S and S' . From these equations we have

$$\begin{aligned} dy'^2 + dz'^2 &= dx^2 + dy^2 + dz^2 + 2\epsilon(dx dx_1 + dy dy_1 + dz dz_1) \\ &\quad + \epsilon^2(dx_1^2 + dy_1^2 + dz_1^2) \end{aligned}$$

which functions satisfy the condition

$$dx dx_1 + dy dy_1 + dz dz_1 = 0,$$

since, for corresponding small lengths on S and S' are equal to within terms of second order in ϵ . When ϵ is taken so small that ϵ^2 may be neglected, the surface S' defined by (1) is said to arise from S by an *infinitesimal deformation* of the latter. In such a deformation every point of S undergoes a displacement along the line through it whose direction-cosines are proportional to x_1, y_1, z_1 . These lines are called the *generatrices* of the deformation.

It is evident that the problem of infinitesimal deformation is equivalent to the solution of equation (2). Since x_1, y_1, z_1 are

functions of u and v , they may be taken for the coordinates of surface S_1 . Equation (2) expresses the fact that the tangent any curve on S is perpendicular to the tangent to the corresponding curve on S_1 at the homologous point. We say that in this case S and S_1 correspond with orthogonality of corresponding linear elements. And so we have

The problem of the infinitesimal deformation of a surface S equivalent to the determination of the surfaces corresponding to with orthogonality of linear elements

153 Characteristic function We proceed to the determination of these surfaces S_1 , and to this end replace equation (2) by the equivalent system

$$(3) \quad \sum \frac{\partial x}{\partial u} \frac{\partial x_1}{\partial u} = 0, \quad \sum \frac{\partial x}{\partial v} \frac{\partial x_1}{\partial v} = 0, \quad \sum \frac{\partial x}{\partial u} \frac{\partial x_1}{\partial v} + \sum \frac{\partial x}{\partial v} \frac{\partial x_1}{\partial u} = 0$$

Weingarten* replaced the last of these equations by the two

$$(4) \quad \sum \frac{\partial x}{\partial u} \frac{\partial x_1}{\partial v} = \phi H, \quad \sum \frac{\partial x}{\partial v} \frac{\partial x_1}{\partial u} = -\phi H,$$

thus defining a function ϕ , which Bianchi has called the *characteristic function*, as usual $H = \sqrt{EG - F^2}$

If the first of equations (3) be differentiated with respect to v and the second with respect to u , we have

$$\sum \frac{\partial x}{\partial u} \frac{\partial^2 x_1}{\partial u \partial v} + \sum \frac{\partial x_1}{\partial u} \frac{\partial^2 x}{\partial u \partial v} = 0, \quad \sum \frac{\partial x}{\partial v} \frac{\partial^2 x_1}{\partial v \partial u} + \sum \frac{\partial x_1}{\partial v} \frac{\partial^2 x}{\partial v \partial u} = 0$$

With the aid of these identities, of the formulas (V, 3), and of the Gauss equations (V, 7), the equations obtained by the differentiation of equations (4) with respect to u and v respectively are reducible to

$$\frac{\partial \phi}{\partial u} = \frac{D \sum \lambda \frac{\partial x_1}{\partial v} - D' \sum \lambda \frac{\partial x}{\partial u}}{H},$$

$$\frac{\partial \phi}{\partial v} = \frac{D' \sum \lambda \frac{\partial x_1}{\partial v} - D'' \sum \lambda \frac{\partial x}{\partial u}}{H}$$

* Crelle, Vol. C (1887), pp. 296-310

ing the case where S is a developable surface, we solve these for $\sum X \frac{\partial x_1}{\partial u}$, $\sum X \frac{\partial x_1}{\partial v}$ and obtain

$$\sum X \frac{\partial x_1}{\partial u} = \frac{D' \frac{\partial \phi}{\partial u} - D \frac{\partial \phi}{\partial v}}{KH}, \quad \sum X \frac{\partial x_1}{\partial v} = \frac{D'' \frac{\partial \phi}{\partial u} - D' \frac{\partial \phi}{\partial v}}{KH},$$

denotes the total curvature of S . If we solve equations and (5) for the derivatives of x_1, y_1, z_1 with respect to u we obtain

$$\begin{cases} \frac{\partial x_1}{\partial u} = \frac{D \left(\phi \frac{\partial X}{\partial v} - X \frac{\partial \phi}{\partial v} \right) - D' \left(\phi \frac{\partial X}{\partial u} - X \frac{\partial \phi}{\partial u} \right)}{KH}, \\ \frac{\partial x_1}{\partial v} = \frac{D' \left(\phi \frac{\partial X}{\partial v} - X \frac{\partial \phi}{\partial v} \right) - D'' \left(\phi \frac{\partial X}{\partial u} - X \frac{\partial \phi}{\partial u} \right)}{KH}, \end{cases}$$

lar expressions in y_1 and z_1 . Hence, when the characteristic is known, the surface S_1 can be obtained by quadratures. The problem reduces therefore to the determination of ϕ . Equations (5) be differentiated with respect to v and u respectively and the resulting equations be subtracted from one another,

$$\frac{\frac{\partial \phi}{\partial v} - D' \frac{\partial \phi}{\partial u}}{KH} + \frac{\partial}{\partial u} \left(\frac{D'' \frac{\partial \phi}{\partial u} - D' \frac{\partial \phi}{\partial v}}{KH} \right) = \sum \frac{\partial x_1}{\partial v} \frac{\partial X}{\partial u} - \sum \frac{\partial x_1}{\partial u} \frac{\partial X}{\partial v}$$

the derivatives of λ, Y, Z in the right-hand member are by the expressions (V, 8), the above equation reduces to

$$\frac{\frac{\partial \phi}{\partial v} - D' \frac{\partial \phi}{\partial u}}{KH} + \frac{\partial}{\partial u} \left(\frac{D'' \frac{\partial \phi}{\partial u} - D' \frac{\partial \phi}{\partial v}}{KH} \right) = \frac{2HD' - FD'' - GD}{H} \phi$$

calls this the *characteristic equation*

sequence of (IV, 73, 74) equation (7) is reducible to

$$\frac{\frac{\partial \phi}{\partial v} - D' \frac{\partial \phi}{\partial u}}{KH} + \frac{\partial}{\partial u} \left(\frac{D'' \frac{\partial \phi}{\partial u} - D' \frac{\partial \phi}{\partial v}}{KH} \right) = \frac{2HD' - FD'' - GD}{KH} \phi,$$

where $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are the coefficients of the linear element of the spherical representation of S , namely

$$(9) \quad d\sigma^2 = \mathcal{E} du^2 + 2\mathcal{F} du dv + \mathcal{G} dv^2$$

$$\text{and} \quad \mathcal{H} = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2}$$

By means of (V, 27) equation (8) is reducible to

$$(10) \quad D'' \left(\frac{\partial^2 \phi}{\partial u^2} - \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}' \frac{\partial \phi}{\partial u} - \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' \frac{\partial \phi}{\partial v} + \mathcal{E} \phi \right) \\ + D \left(\frac{\partial^2 \phi}{\partial v^2} - \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' \frac{\partial \phi}{\partial u} - \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}' \frac{\partial \phi}{\partial v} + \mathcal{G} \phi \right) \\ - 2D' \left(\frac{\partial^2 \phi}{\partial u \partial v} - \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \frac{\partial \phi}{\partial u} - \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \frac{\partial \phi}{\partial v} + \mathcal{F} \phi \right) = 0,$$

where the Christoffel symbols are formed with respect to (9)

Since X , Y , Z are solutions of equations (V, 22), they are solutions of (10), and consequently also of equation (7). Therefore the latter equation may be written

$$\frac{\partial}{\partial v} \left[\frac{D \left(X \frac{\partial \phi}{\partial v} - \phi \frac{\partial X}{\partial v} \right) - D' \left(X \frac{\partial \phi}{\partial u} - \phi \frac{\partial X}{\partial u} \right)}{KH} \right] \\ + \frac{\partial}{\partial u} \left[\frac{D'' \left(X \frac{\partial \phi}{\partial u} - \phi \frac{\partial X}{\partial u} \right) - D' \left(X \frac{\partial \phi}{\partial v} - \phi \frac{\partial X}{\partial v} \right)}{KH} \right] = 0$$

But this is the condition of integrability of equations (6). Hence we have the theorem

Each solution of the characteristic equation determines a surface S_1 and consequently an infinitesimal deformation of S

154 Asymptotic lines parametric When the asymptotic line on S are parametric, equation (10) is reducible, in consequence of (VI, 15), to

$$(11) \quad \frac{\partial^2 \phi}{\partial u \partial v} + \frac{1}{2} \frac{\partial \log \rho}{\partial v} \frac{\partial \phi}{\partial u} + \frac{1}{2} \frac{\partial \log \rho}{\partial u} \frac{\partial \phi}{\partial v} + \mathcal{F} \phi = 0,$$

where

$$\frac{1}{\rho} = \sqrt{-K} = \frac{\mathcal{H}}{D'}$$

If we put

$$\phi \sqrt{-\epsilon \rho} = \theta,$$

+1 or -1 according as the curvature of S is positive or negative, equation (11) becomes

$$\frac{\partial^2 \theta}{\partial u \partial v} = \left(\frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial u \partial v} - \mathcal{F} \right) \theta$$

Y, Z are solutions of (11), the functions

$$\nu_1 = X\sqrt{-\epsilon\rho}, \quad \nu_2 = Y\sqrt{-\epsilon\rho}, \quad \nu_3 = Z\sqrt{-\epsilon\rho}$$

solutions of (12)

equations (6) may be put in the form

$$\frac{\partial x_1}{\partial u} = - \begin{vmatrix} \nu_1 & \theta \\ \frac{\partial \nu_1}{\partial u} & \frac{\partial \theta}{\partial u} \end{vmatrix}, \quad \frac{\partial x_1}{\partial v} = \begin{vmatrix} \nu_1 & \theta \\ \frac{\partial \nu_1}{\partial v} & \frac{\partial \theta}{\partial v} \end{vmatrix}$$

we should compare these equations with the Leleuvre's (§ 79), which give the expressions for the derivatives of the coordinates of S in terms of ν_1, ν_2, ν_3

from these results it follows that any three solutions of (11) of the form

$$\frac{\partial^2 \theta}{\partial u \partial v} = M\theta,$$

if M is any function of u and v , determine a surface S upon which the parametric curves are the asymptotic lines, and every deformation linearly independent of these three gives by quadrature an infinitesimal deformation of S

EXAMPLES

A necessary and sufficient condition that two surfaces satisfying the conditions applicable is that they be minimal surfaces adjoint to one another

x, y, z and x_1, y_1, z_1 satisfy the condition (2), so also do ξ, η, ζ and ξ_1, η_1, ζ_1 , the latter being given by

$$\xi = a_1x + b_1y + c_1z + d_1, \quad x_1 = a_1\xi_1 + a_2\eta_1 + a_3\zeta_1 + e_1,$$

$$\eta = a_2x + b_2y + c_2z + d_2, \quad y_1 = b_1\xi_1 + b_2\eta_1 + b_3\zeta_1 + e,$$

$$\zeta = a_3x + b_3y + c_3z + d_3, \quad z_1 = c_1\xi_1 + c_2\eta_1 + c_3\zeta_1 + e_3,$$

a_2, a_3, e_1, e_2, e_3 are constants

A necessary condition that the locus of the point (x_1, y_1, z_1) be a curve is that the surface be developable. In this case any orthogonal trajectory of the lines to S satisfies the condition

to investigate the cases $\phi = 0$ and $\phi = c$, where c is a constant different from zero

S_1 and S_2 correspond to S with orthogonality of linear elements, so also the locus of a point dividing in constant ratio the line joining corresponding S_1 and S_2

155 Associate surfaces The expressions in the parentheses of equation (10) differ only in sign from the second fundamental coefficients, D_0 , D'_0 , D''_0 , of the surface S_0 enveloped by the plane

$$(14) \quad Xx + Yy + Zz = \phi^*$$

Hence equation (10) may be written

$$(15) \quad D''D_0 + DD''_0 - 2D'D'_0 = 0$$

This is the condition that to the asymptotic lines upon either of the surfaces S , S_0 there corresponds a conjugate system on the other (§ 56). Bianchi applies the term *associate* to two surfaces whose tangent planes at corresponding points are parallel and for which the asymptotic lines on either correspond to a conjugate system on the other. Since the converse of the preceding results are readily shown to be true, we have the theorem of Bianchi †

When two surfaces are associate the expression for the distance from a fixed point in space to the tangent plane to one is the characteristic function for an infinitesimal deformation of the other

Hence the problems of infinitesimal deformation and of the determination of surfaces associate to a given one are equivalent. We consider the latter problem.

Since the tangent planes to S and S_0 at corresponding points are parallel, we have

$$\frac{\partial x_0}{\partial u} = \lambda \frac{\partial x}{\partial u} - \mu \frac{\partial x}{\partial v}, \quad \frac{\partial x_0}{\partial v} = \sigma \frac{\partial x}{\partial u} - \tau \frac{\partial x}{\partial v},$$

and similar equations in y_0 and z_0 , where λ , μ , σ , τ are functions of u and v to be determined ‡

If these equations be multiplied by $\frac{\partial X}{\partial u}$, $\frac{\partial Y}{\partial u}$, $\frac{\partial Z}{\partial u}$ and added, and likewise by $\frac{\partial X}{\partial v}$, $\frac{\partial Y}{\partial v}$, $\frac{\partial Z}{\partial v}$ and added, we obtain

$$(16) \quad \begin{cases} D_0 = \lambda D - \mu D', & D''_0 = \sigma D' - \tau D'', \\ D'_0 = \lambda D' - \mu D'' = \sigma D - \tau D', \end{cases}$$

* Cf. § 67

† *Lezioni*, Vol. II, p. 9

‡ The negative signs before μ and τ are taken so that subsequent results may have suitable form

, D'_0, D''_0 are the second fundamental quantities for S_0 . These values are substituted in (15), we find

$$\lambda - \tau = 0$$

hence the above equations reduce to

$$\frac{\partial x_0}{\partial u} = \lambda \frac{\partial x}{\partial u} - \mu \frac{\partial x}{\partial v}, \quad \frac{\partial x_0}{\partial v} = \sigma \frac{\partial x}{\partial u} - \lambda \frac{\partial x}{\partial v}$$

On the use of the Gauss equations (V, 7), the condition of integrability of equations (18) is reducible to

$$A \frac{\partial x}{\partial u} + B \frac{\partial x}{\partial v} = 0,$$

where A and B are determinate functions. Since similar equations hold for y and z , both A and B must be identically zero. Calculating the coefficients for these functions, we have the following equations satisfied by λ, μ , and σ

$$\begin{cases} \frac{\partial \mu}{\partial v} - \frac{\partial \lambda}{\partial u} + \mu \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} - 2\lambda \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} + \sigma \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} = 0, \\ \frac{\partial \sigma}{\partial u} - \frac{\partial \lambda}{\partial v} + \mu \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} - 2\lambda \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + \sigma \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} = 0 \end{cases}$$

In these equations we must add

$$2\lambda D' - \mu D'' - \sigma D = 0,$$

from the last of (16). The determination of the associates of a given surface referred to any parametric system involves the integration of this system of equations. Moreover, the existence of solutions leads to an associate surface. We shall now consider several cases in which the parametric curves are of a particular kind.

Particular parametric curves. Suppose that S is a surface in which the parametric curves form a conjugate system. We are under what conditions there exists an associate surface S' in which also the corresponding curves form a conjugate

On this hypothesis we have, from (16),

$$\mu = \sigma = 0,$$

so that equations (19) reduce to

$$(21) \quad \frac{\partial \log \lambda}{\partial u} = -2 \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}, \quad \frac{\partial \log \lambda}{\partial v} = -2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\},$$

which are consistent only when

$$(22) \quad \frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} = \frac{\partial}{\partial v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\},$$

that is, when the point equation of S , namely

$$\frac{\partial^2 \theta}{\partial u \partial v} = \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} \frac{\partial \theta}{\partial u} + \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} \frac{\partial \theta}{\partial v},$$

has equal invariants (cf § 165)

Conversely, when condition (22) is satisfied, the function given by the quadratures (21) makes the equations

$$(23) \quad \frac{\partial x_0}{\partial u} = \lambda \frac{\partial x}{\partial u}, \quad \frac{\partial x_0}{\partial v} = -\lambda \frac{\partial x}{\partial v}$$

compatible, and thus the coordinates of an associate surface are obtained by quadratures. Hence we have the theorem of Cosserat*

The infinitesimal deformation of a surface S is the same problem as the determination of the conjugate systems with equal point invariants on S

Since the relation between S and S_0 is reciprocal and the parametric curves are conjugate for both surfaces, these curves on S_0 also have equal point invariants

If S be referred to its asymptotic lines, the corresponding lines on S_0 form a conjugate system. In this case, as is seen from (16) λ is zero and equations (18) reduce to

$$(24) \quad \frac{\partial x_0}{\partial u} = -\mu \frac{\partial x}{\partial v}, \quad \frac{\partial x_0}{\partial v} = \sigma \frac{\partial x}{\partial u},$$

moreover, equations (19) become

$$(25) \quad \begin{cases} \frac{\partial \mu}{\partial v} + \mu \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} + \sigma \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} = 0, \\ \frac{\partial \sigma}{\partial u} + \mu \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} + \sigma \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\} = 0 \end{cases}$$

olution of this system is the same problem as the integration of a partial differential equation of the second order, as is the elimination of either unknown. When a solution of one is obtained, the corresponding value of the other is given directly by one of equations (25).

Make an application of these results to a ruled surface, we suppose to be referred to its asymptotic lines. If the $u = \text{const}$ are the generators, they are geodesics, and consequently (VI, 50)

$$\begin{Bmatrix} 11 \\ 2 \end{Bmatrix} = 0$$

u can be found by a quadrature. When this value is substituted in the second of equations (25), we have a linear equation, and consequently σ also can be obtained by quadratures. We have the theorem

the curved asymptotic lines on a ruled surface are known, the surfaces can be found by quadratures

were referred to its asymptotic lines, we should have equations similar to (24). These equations may be interpreted as

tangent to an asymptotic line on one of two associate surfaces is parallel to the direction conjugate to the corresponding curve on the other surface

EXAMPLES

If two associate surfaces are applicable to one another, they are minimal

Every surface of translation admits an associate surface of translation such that the generatrices of the two surfaces constitute the common conjugate system

Surfaces associate to a sphere are minimal

In the equations of the right helicoid we

$$x = u \cos v, \quad y = u \sin v, \quad z = av,$$

the characteristic function of any infinitesimal deformation is $\phi = (U + V)(u^2 + v^2)^{-\frac{1}{2}}$, where U and V are arbitrary functions of u and v respectively. Find the surfaces which admit such deformations and show that the latter are molding surfaces.

If S_0 and S_1 are associate surfaces of a surface S , the locus of a point which divides the joins of corresponding points of S_0 and S_1 in a constant ratio is a surface of S .

157 Relations between three surfaces S, S_1, S_0 Having thus discussed the various ways in which the problem of infinitesimal deformation may be attacked, we proceed to the consideration of other properties which are possessed by a set of three surfaces S, S_1, S_0

We recall the differential equation

$$dx dx_1 + dy dy_1 + dz dz_1 = 0,$$

and remark that it may be replaced by the three

$$(26) \quad dx_1 = z_0 dy - y_0 dz, \quad dy_1 = x_0 dz - z_0 dx, \quad dz_1 = y_0 dx - x_0 dy,$$

if the functions x_0, y_0, z_0 are such a form that the conditions of integrability of equations (26) are satisfied. These conditions are

$$\begin{aligned} \frac{\partial y}{\partial u} \frac{\partial z_0}{\partial v} - \frac{\partial z}{\partial u} \frac{\partial y_0}{\partial v} &= \frac{\partial y}{\partial v} \frac{\partial z_0}{\partial u} - \frac{\partial z}{\partial v} \frac{\partial y_0}{\partial u}, \\ \frac{\partial z}{\partial u} \frac{\partial x_0}{\partial v} - \frac{\partial x}{\partial u} \frac{\partial z_0}{\partial v} &= \frac{\partial z}{\partial v} \frac{\partial x_0}{\partial u} - \frac{\partial x}{\partial v} \frac{\partial z_0}{\partial u}, \\ \frac{\partial x}{\partial u} \frac{\partial y_0}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x_0}{\partial v} &= \frac{\partial x}{\partial v} \frac{\partial y_0}{\partial u} - \frac{\partial y}{\partial v} \frac{\partial x_0}{\partial u} \end{aligned}$$

If these equations be multiplied by $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}$ respectively and added, and likewise by $\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}$, and by X, Y, Z , we obtain, by (IV, 2),

$$(27) \quad \sum X \frac{\partial x_0}{\partial u} = 0, \quad \sum X \frac{\partial x_0}{\partial v} = 0,$$

$$(28) \quad \begin{vmatrix} X & Y & Z \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x_0}{\partial v} & \frac{\partial y_0}{\partial v} & \frac{\partial z_0}{\partial v} \end{vmatrix} = \begin{vmatrix} X & Y & Z \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x_0}{\partial u} & \frac{\partial y_0}{\partial u} & \frac{\partial z_0}{\partial u} \end{vmatrix}.$$

From the first two of these equations it follows that the locus of the point with coordinates x_0, y_0, z_0 corresponds to S with parallelism of tangent planes

In order to interpret the last of these equations we recall from § 61 that

$$X = \frac{a}{h} \frac{\partial(Y, Z)}{\partial(u, v)}, \quad Y = \frac{a}{h} \frac{\partial(Z, X)}{\partial(u, v)}, \quad Z = \frac{a}{h} \frac{\partial(X, Y)}{\partial(u, v)},$$

α is ± 1 according as the curvature of the surface is positive or negative. If we substitute these values in the left-hand members of the following equations, and add and subtract $\frac{\partial x}{\partial u} \frac{\partial X}{\partial u} \frac{\partial X}{\partial v}$ from these equations respectively, the resulting equations are reducible to the form of the right-hand members

$$\begin{cases} X \left(Y \frac{\partial z}{\partial u} - Z \frac{\partial y}{\partial u} \right) = \alpha \left(D' \frac{\partial X}{\partial u} - D \frac{\partial X}{\partial v} \right), \\ X \left(Y \frac{\partial z}{\partial v} - Z \frac{\partial y}{\partial v} \right) = \alpha \left(D'' \frac{\partial X}{\partial u} - D' \frac{\partial X}{\partial v} \right) \end{cases}$$

and of these and similar identities, equation (28) can be reduced into

$$\sum \frac{\partial x_0}{\partial u} \frac{\partial X}{\partial u} + D \sum \frac{\partial x_0}{\partial v} \frac{\partial X}{\partial v} - D' \left(\sum \frac{\partial x_0}{\partial u} \frac{\partial X}{\partial v} + \sum \frac{\partial x_0}{\partial v} \frac{\partial X}{\partial u} \right) = 0$$

this equation is equivalent to (15) because of (27), the coordinates x_0, y_0, z_0 in (26) are the coordinates of S_0 . Hence if a surface S_1 is known, the coordinates of the corresponding surface S_0 are readily found.

This result enables us to find another property of S_0 and S_1 . If Y_1, Z_1 denote the direction-cosines of the normal to S_1 , we are given by

$$Y_1 = \frac{1}{H_1} \frac{\partial(y_1, z_1)}{\partial(u, v)}, \quad X_1 = \frac{1}{H_1} \frac{\partial(z_1, x_1)}{\partial(u, v)}, \quad Z_1 = \frac{1}{H_1} \frac{\partial(x_1, y_1)}{\partial(u, v)},$$

where $H_1 = \sqrt{E_1 G_1 - F_1^2}$, E_1, F_1, G_1 being the coefficients of the first fundamental form of S_1 . If the values of the derivatives of x_1, y_1, z_1 given by (26), be substituted in these expressions, we have, in consequence of (14),

$$X_1 = \frac{x_0 H}{H_1} \phi, \quad Y_1 = \frac{y_0 H}{H_1} \phi, \quad Z_1 = \frac{z_0 H}{H_1} \phi$$

as an immediate consequence we have the theorem

The normal to S_1 is parallel to the radius vector of S_0 at the corresponding point.

By means of (30) we find readily the expressions for the second fundamental coefficients D_1 , D'_1 , D''_1 of S_1 . If we notice that

$$\sum x_0 \frac{\partial x_1}{\partial u} = 0, \quad \sum x_0 \frac{\partial x_1}{\partial v} = 0,$$

and substitute the values from (6) and (30) in

$$D_1 = -\sum \frac{\partial x_1}{\partial u} \frac{\partial X_1}{\partial u}, \quad D'_1 = -\sum \frac{\partial x_1}{\partial u} \frac{\partial X_1}{\partial v} = -\sum \frac{\partial x_1}{\partial v} \frac{\partial X_1}{\partial u},$$

$$D''_1 = -\sum \frac{\partial x_1}{\partial v} \frac{\partial X_1}{\partial v},$$

we obtain

$$(31) \quad \begin{cases} D_1 = \frac{\phi^2}{H_1 K} (DD'_0 - D'D_0), \\ D'_1 = \frac{\phi^2}{H_1 K} (DD''_0 - D'D'_0) = \frac{\phi^2}{H_1 K} (D'D'_0 - D''D_0), \\ D''_1 = \frac{\phi^2}{H_1 K} (D'D''_0 - D''D'_0) \end{cases}$$

From these expressions follow

$$(32) \quad \begin{cases} D_1 D'' + D'_1 D - 2 D'_1 D' = 0, \\ D_1 D''_0 + D'_1 D_0 - 2 D'_1 D'_0 = 0 \end{cases}$$

Combining this result with (15), we have

The asymptotic lines upon any one of a group of three surfaces S , S_1 , S_0 correspond to a conjugate system on the other two,

or, in other words

The system of lines which is conjugate for any two of three surfaces S , S_1 , S_0 corresponds to the asymptotic lines on the other

If the curvature of S be negative, its asymptotic lines are real, and consequently the common conjugate system on S_1 and S_0 is real. If these lines be parametric, the second of equations (32) reduces to

$$D_1 D''_0 + D'_1 D_0 = 0$$

As an odd number of the four quantities in this equation must be negative, either S_0 or S_1 has positive curvature and the other negative. Similar results follow if we begin with the assumption that S_1 or S_0 has negative curvature.

If the curvature of S be positive, the conjugate system common to it and S_1 is real (cf § 56), consequently the asymptotic lines

is real, and the curvature of the latter is negative. But that when the curvature of S_0 is negative, and of S positive, S_1 also is negative. Hence

a set of three surfaces S , S_1 , S_0 , one and only one of them have curvature

use that S is referred to the conjugate system corresponding to the asymptotic lines on S_0 . The point equation of S is

$$\frac{\partial^2 \theta}{\partial u \partial v} - \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} \frac{\partial \theta}{\partial u} - \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} \frac{\partial \theta}{\partial v} = 0$$

I prove that this is the point equation of S_1 also. I differentiate the equation

$$\frac{\partial x_1}{\partial u} = z_0 \frac{\partial y}{\partial u} - y_0 \frac{\partial z}{\partial u}$$

with respect to v , and make use of the fact that y and z are solutions of (33), we have, in consequence of (26),

$$\frac{\partial^2 x_1}{\partial u \partial v} = \left(\frac{\partial z_0}{\partial v} \frac{\partial y}{\partial u} - \frac{\partial y_0}{\partial v} \frac{\partial z}{\partial u} \right) + \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} \frac{\partial x_1}{\partial u} + \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} \frac{\partial x_1}{\partial v}$$

The expression in parenthesis is zero in consequence of equation (24), and hence x_1 is a solution of (33).

The parametric curves on S_0 are its asymptotic lines, the representation of S_0 and consequently of S must satisfy the condition

$$\frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' = \frac{\partial}{\partial v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}'$$

We have the theorem of Cosserat

problem of infinitesimal deformation of a surface is the same as the determination of the conjugate systems with equal tangential curvatures upon the surface

Surfaces resulting from an infinitesimal deformation We consider the consideration of the surface S' arising from an infinitesimal deformation of S . Its coordinates are given by

$$x' = x + \epsilon x_1, \quad y' = y + \epsilon y_1, \quad z' = z + \epsilon z_1,$$

where ϵ is a small constant whose powers higher than the first are neglected. Since the fundamental quantities of the first order for S' are equal to the corresponding ones for S , by

means of (26) the expressions for the direction-cosines X' , Y' , Z' of the normal to S' are reducible to

$$(35) \quad X' = X + \epsilon(Yz_0 - Zy_0)$$

and similar expressions for Y' and Z'

The derivatives of X' with respect to u and v are reducible by means of (29) to

$$\begin{aligned} \frac{\partial X'}{\partial u} &= \frac{\partial X}{\partial u} + \epsilon \left(z_0 \frac{\partial Y}{\partial u} - y_0 \frac{\partial Z}{\partial u} \right) + \frac{\epsilon a}{\mathcal{H}} \left(D'_0 \frac{\partial X}{\partial u} - D_0 \frac{\partial X}{\partial v} \right), \\ \frac{\partial X'}{\partial v} &= \frac{\partial X}{\partial v} + \epsilon \left(z_0 \frac{\partial Y}{\partial v} - y_0 \frac{\partial Z}{\partial v} \right) + \frac{\epsilon a}{\mathcal{H}} \left(D''_0 \frac{\partial X}{\partial u} - D'_0 \frac{\partial X}{\partial v} \right), \end{aligned}$$

where a is ± 1 according as the curvature of S_0 is positive or negative. When these results are combined with (26) and (34), we obtain

$$\begin{aligned} \sum \frac{\partial x'}{\partial u} \frac{\partial X'}{\partial u} &= \sum \frac{\partial x}{\partial u} \frac{\partial X}{\partial u} + \frac{\epsilon a}{\mathcal{H}} \sum \frac{\partial x}{\partial u} \left(D'_0 \frac{\partial X}{\partial u} - D_0 \frac{\partial X}{\partial v} \right) \\ &\quad + \epsilon \sum \left[z_0 \left(\frac{\partial X}{\partial u} \frac{\partial y}{\partial u} + \frac{\partial Y}{\partial u} \frac{\partial x}{\partial u} \right) - y_0 \left(\frac{\partial X}{\partial u} \frac{\partial z}{\partial u} + \frac{\partial Z}{\partial u} \frac{\partial x}{\partial u} \right) \right] \end{aligned}$$

The last expression is identically zero, as one sees by writing it out in full. From this and similar expressions for $\sum \frac{\partial x'}{\partial u} \frac{\partial X'}{\partial v}$, $\sum \frac{\partial x'}{\partial v} \frac{\partial X'}{\partial u}$, and $\sum \frac{\partial x'}{\partial v} \frac{\partial X'}{\partial v}$, the values for the second fundamental coefficients of S' can be given in the form

$$(36) \quad \begin{cases} \mathcal{A} = - \sum \frac{\partial x'}{\partial u} \frac{\partial X'}{\partial u} = D + \frac{\epsilon a}{\mathcal{H}} (D'_0 D - D_0 D'), \\ \mathcal{A}' = D' + \frac{\epsilon a}{\mathcal{H}} (D''_0 D - D'_0 D') = D' + \frac{\epsilon a}{\mathcal{H}} (D'_0 D' - D_0 D''), \\ \mathcal{A}'' = D'' + \frac{\epsilon a}{\mathcal{H}} (D'_0 D' - D'_0 D'') \end{cases}$$

We know that $\mathcal{H}$ is equal to $\pm HK$ according as the curvature of S is positive or negative (cf § 60). Also, by § 157, one and only one of three surfaces S , S_1 , S_0 has positive curvature. Recalling that a in the above formulas is ± 1 according as the curvature of S_0 is positive or negative, we can, in consequence of (31), write equations (36) in the form

$$(37) \quad \mathcal{A} = D \pm \epsilon \frac{H_1}{\phi^2 H} D_1, \quad \mathcal{A}' = D' \pm \epsilon \frac{H_1}{\phi^2 H} D'_1, \quad \mathcal{A}'' = D'' \pm \epsilon \frac{H_1}{\phi^2 H} D''_1,$$

where the upper sign holds when S_1 has positive curvature

these equations it is seen that $\mathcal{A}'$ and D' can be zero simultaneously only when D'_1 is zero. Hence we have

unique conjugate system which remains conjugate in an infinitesimal deformation of a surface is the one corresponding to a conjugate system on S_1 , or, what is the same thing, to the asymptotic lines

particular, in order that the curves of this conjugate system be lines of curvature, it is necessary and sufficient that the l representation be orthogonal, and consequently that S_0 be a minimal surface (cf § 55). From this it follows that the spherical representation of the lines of curvature of S is isothermal. Conversely, if a surface is of this kind, there is a unique minimal surface with the same representation of its asymptotic lines, and this can be found by quadratures. Hence the required infinitesimal deformation of the given surface can be effected by quadratures (6), and so we have the theorem of Weingarten *

necessary and sufficient condition that a surface admit an infinitesimal deformation which preserves its lines of curvature is that the l representation of the latter be isothermal, when such a deformation is expressed in terms of parameters referring to its lines of curvature, the deformation can be effected by quadratures

Isothermic surfaces By means of the results of § 158 we obtain an important theorem concerning surfaces whose lines of curvature form an isothermal system. They are called *isothermic* (cf Exs 1, 3, p 159).

From equations (23) it follows that if the common conjugate system on two associate surfaces is orthogonal for one it is the same for the other. In this case equation (22) reduces to

$$\frac{\sigma}{\partial u \partial v} \log \frac{E}{G} = 0,$$

hence the general integral is

$$\frac{E}{G} = \frac{U}{V},$$

where U and V are functions of u and v respectively. Hence the lines of curvature on S form an isothermal system (cf § 41).

* *Sitzungsberichte der Königlich Akademien zu Berlin*, 1886

If the parameters be isothermic and the linear element written

$$ds^2 = r(du^2 + dv^2),$$

it follows from (21) that

$$(37) \quad \lambda = \frac{1}{r},$$

and equations (23) become

$$\frac{\partial x_0}{\partial u} = \frac{1}{r} \frac{\partial x}{\partial u}, \quad \frac{\partial x_0}{\partial v} = -\frac{1}{r} \frac{\partial x}{\partial v}$$

From these results we derive the following theorem of Bour* and Christoffel

If the linear element of an isothermic surface referred to its lines of curvature be

$$ds^2 = r(du^2 + dv^2),$$

a second isothermic surface can be found by quadratures It is associate to the given one, and its linear element is

$$ds_0^2 = \frac{1}{r}(du^2 + dv^2)$$

From equations (16) and (17) it follows that the equation of the common conjugate system (IV, 43) on two associate surfaces S, S_0 is reducible to

$$(38) \quad \mu du^2 + 2\lambda dudv + \sigma dv^2 = 0$$

The preceding results tell us that a necessary and sufficient condition that S be an isothermic surface is that there be a set of solutions of equations (19) such that (38) is the equation of the lines of curvature on S . Hence there must be a function ρ such that

$$\mu = \rho(ED' - FD), \quad 2\lambda = \rho(ED'' - GD), \quad \sigma = \rho(FD'' - GD')$$

satisfy equations (19) † Upon substitution we are brought to two equations of the form

$$\frac{\partial \log \rho}{\partial u} = \alpha, \quad \frac{\partial \log \rho}{\partial v} = \beta,$$

where α and β are determinate functions of u and v . In order that S be isothermic, these functions must satisfy the condition

$$\frac{\partial \alpha}{\partial v} = \frac{\partial \beta}{\partial u}$$

When it is satisfied, ρ and consequently μ, λ, σ are given by quadratures

* *Journal de l'École Polytechnique* Cahier 39 (1862) p. 118

† Cf. Bianchi Vol. II p. 30

der furthermore the form

$$H(\mu du^2 + 2\lambda dudv + \sigma dv^2)$$

(7) it is seen that when the lines of curvature are parathis expression reduces to $2 dudv$. Hence its curvature is $V, 12)$, and consequently the curvature of (89) is zero 135 it follows that this form is reducible to $du_1 dv_1$ by quad- Hence we have the theorem of Weingarten

lines of curvature upon an isothermic surface can be found by res

clude this discussion of isothermic surfaces with the proof of a theorem our He introduced the term *limit surfaces* of a group of applicable sur- designate the members of the group whose mean curvature is a maximum im According to Ribaucour,

ut surfaces of a group of applicable surfaces are isothermic

ving it we consider a member S of the group referred to its lines of cur- its mean curvature is given by $D/E + D'/G$. In consequence of equations lean curvature of a near-by surface is, to within terms of higher order,

$$\frac{D}{E} + \frac{D'}{G} = \frac{D}{E} + \frac{D'}{G} \pm \frac{\epsilon}{\rho} D_0 \left(\frac{D}{E} - \frac{D'}{G} \right)$$

ry and sufficient condition that the mean curvature of S be a maximum im is consequently

$$D_0 \left(\frac{D}{E} - \frac{D'}{G} \right) = 0$$

the case of the sphere for which the expression in parenthesis is zero, at D_0 is zero. Hence the common conjugate system of S and S_0 is com- lines of curvature on the former, and therefore S is isothermic

GENERAL EXAMPLES

y, z and x_1, y_1, z_1 are the coordinates of two surfaces corresponding with lity of linear elements, the coordinates of a pair of applicable surfaces by

$$\begin{aligned} \xi_1 &= x + tx_1, & \eta_1 &= y + ty_1, & \zeta_1 &= z + tz_1, \\ \xi_2 &= x - tx_1, & \eta_2 &= y - ty_1, & \zeta &= z - tz_1, \end{aligned}$$

any constant

vo surfaces are applicable, the locus of the mid point of the line joining ling points admits of an infinitesimal deformation in which this line is tix

atever be the surface S , the characteristic equation (7) admits the solu $X + bY + cZ$, where a, b, c are constants. Show that S_0 is the point id that equations (26) become

$$x_1 = cy - bz + d, \quad y_1 = az - cx + e, \quad z_1 = bx - ay + f,$$

, f are constants, that consequently S_1 is a plane, and that the infinitesi- nation is in reality an infinitesimal displacement

4 Determine the form of the results of Exs 1, 2, where ϕ has the value of Ex 3

5 Show that the first fundamental coefficients E_1, F_1, G_1 of a surface S_1 are of the form

$$E_1 = E\phi^2 + \frac{1}{H^2K^2} \left(D \frac{\partial \phi}{\partial v} - D' \frac{\partial \phi}{\partial u} \right)^2,$$

$$F_1 = F\phi^2 + \frac{1}{H^2K^2} \left(D \frac{\partial \phi}{\partial v} - D' \frac{\partial \phi}{\partial u} \right) \left(D' \frac{\partial \phi}{\partial v} - D'' \frac{\partial \phi}{\partial u} \right),$$

$$G_1 = G\phi^2 + \frac{1}{H^2K^2} \left(D' \frac{\partial \phi}{\partial v} - D'' \frac{\partial \phi}{\partial u} \right)^2$$

6 Let Σ denote the locus of the point which bisects the segment of the normal to a surface S between the centers of principal curvature of the latter. In order that the lines on Σ corresponding to the lines of curvature on S shall form a conjugate system, it is necessary and sufficient that Σ correspond to a minimal surface with orthogonality of linear elements, and that the latter surface and S correspond with parallelism of tangent planes

7 Show that when the spherical representation of the asymptotic lines of a surface is in the condition

$$\frac{\partial}{\partial u} \left\{ \frac{11}{2} \right\}' = \frac{\partial}{\partial v} \left\{ \frac{22}{1} \right\}',$$

the lines (25) admit two pairs of solutions which are such that $\mu = \sigma$ and $\mu = -\sigma$. The two associate surfaces S_0, S'_0 thus found by quadratures the parameters of which are isothermal-conjugate, and S_0 and S'_0 are associates of one another

8 Show that the equation of Ex 7 is a necessary and sufficient condition that the two surfaces associate to S be associate to one another

9 Show that when the sphere is referred to its minimal lines, the condition of Ex 7 is satisfied, and investigate this case

10 On any surface associate to a pseudospherical surface the curves corresponding to the asymptotic lines of the latter are geodesics. A surface with a conjugate system of geodesics is called a *surface of Voss* (cf § 170)

11 Determine whether minimal surfaces and the surfaces associate to pseudospherical surfaces are the only surfaces of Voss

12 When the equations of a central quadric are in the form (VII, 35), the associate surfaces are given by

$$x_0 = \sqrt{a} \left[\int U(1-u^2) du + \int V(1-v^2) dv \right],$$

$$y_0 = 2\sqrt{b} \left[\int Uu du + \int Vv dv \right],$$

$$z_0 = 2\sqrt{c} \left[\int U(1+u^2) du - \int V(1+v^2) dv \right]$$

where U and V are arbitrary functions of u and v respectively. Hence the associate surfaces are surfaces of translation

13 When the equations of a paraboloid are in the form

$$x = \sqrt{a}(u+v), \quad y = \sqrt{b}(u-v), \quad z = 2uv,$$

the associate surfaces are surfaces of translation whose generators are plane curves. Their equations are

$$x_0 = \sqrt{a}(U+V), \quad y_0 = \sqrt{b}(V-U), \quad z_0 = 2 \int uU du + 2 \int vV dv,$$

where U and V are arbitrary functions of u and v respectively

ow that a quadric admits of an infinitesimal deformation which preserves curvature, and determine the corresponding associate surface

ice the relation between S and S_1 is reciprocal, there is a surface S_2 to S_1 which bears to S a relation similar to that of S_0 to S_1 . Show that ptotic lines on S_0 and S_2 correspond, and that these surfaces are polar with respect to the imaginary sphere $x^2 + y^2 + z^2 + 1 = 0$

ice the relation between S and S_0 is reciprocal, there is a surface S_2 cor- g to S_0 with orthogonality of linear elements which bears to S a relation that of S_1 to S_0 . Show that the asymptotic lines on S_1 and S_2 correspond, coordinates of the latter are such that

$$x_1 - x_2 = yz_0 - zy_0, \quad y_1 - y_2 = zx_0 - xz_0, \quad z_1 - z_2 = xy_0 - yx_0,$$

he line joining corresponding points on S_1 and S_2 is tangent to both surfaces

ow that if S_3 denotes the surface corresponding to S_2 with orthogonality elements which is determined by S_1 , associate to S_3 , the surfaces S and ated to one another in a manner similar to S_1 and S_2 of Ex 16

ow that the surface S_4 , which is the associate to S_2 determined by S_0 , is reciprocal of S with respect to the imaginary sphere $x^2 + y^2 + z^2 + 1 = 0$

we continue the process introduced in the foregoing examples, we obtain nces of surfaces

$$\begin{array}{ccccccccccc} S, & S_1, & S_3, & S_5, & S_7, & S_9, & S_{11}, & & & & \\ S, & S_0, & S_2, & S_4, & S_6, & S_8, & S_{10}, & & & & \end{array}$$

t S_{11} and S_{10} are the same surface, likewise S_{12} and S_0 , and that conse- reie is a closed system of twelve surfaces, they are called the *twelve sur- arbourx*

necessary and sufficient condition that a surface referred to its minimal isothermic is that

$$\frac{D}{D} = \frac{U}{V},$$

and V are functions of u and v respectively

necessary and sufficient condition that the lines of curvature on an iso- urface be represented on the sphere by an isothermal system is that

$$\frac{\rho_1}{\rho_2} = \frac{U}{V},$$

and V are functions of u and v respectively, the latter being parameters to the lines of curvature. Show that the parameters of the asymptotic uch surface can be so chosen that $F = G$

ow that an isothermic surface is transformed by an inversion into an surface

S_1 and S_2 are the sheets of the envelope of a family of spheres of two is, which are not orthogonal to a fixed sphere, and the points of contact of e are said to correspond, in order that the correspondence be conformal, ssary that the lines of curvature on S_1 and S_2 correspond and that these e isothermic (cf Ex 15, Chap XIII)

CHAPTER XII

RECTILINEAR CONGRUENCES

160 Definition of a congruence Spherical representation A two-parameter system of straight lines in space is called a *rectilinear congruence*. The normals to a surface constitute such a system, likewise the generatrices of an infinitesimal deformation of a surface (cf § 152). Later we shall find that in general the lines of a congruence are not normal to a surface. Hence congruences of normals form a special class, they are called *normal congruences*. They were the first studied, particularly in investigations of the effects of reflection and refraction upon rays of light. The first purely mathematical treatment of general rectilinear congruences was given by Kummer in his memoir, *Allgemeine Theorie der gradlinigen Strahlensysteme* *. We begin our treatment of the subject with the derivation of certain of Kummer's results by methods similar to his own.

From the definition of a congruence it follows that its lines meet a given plane in such a way that through a point of the plane one line, or at most a finite number, pass. Similar results hold if a surface be taken instead of a plane, this surface is called the *surface of reference*. And so we may define a congruence analytically by means of the coordinates of the latter surface in terms of two parameters u, v , and by the direction cosines of the lines in terms of these parameters. Thus, a congruence is defined by a set of equations such as

$$(1) \quad \begin{cases} x = f_1(u, v), & y = f_2(u, v), & z = f_3(u, v), \\ X = \phi_1(u, v), & Y = \phi_2(u, v), & Z = \phi_3(u, v), \end{cases}$$

where the functions f and ϕ are analytic in the domain of u and v under consideration, and the functions ϕ are such that

$$X^2 + Y^2 + Z^2 = 1$$

* *Crelle* Vol I VII (1860) pp 189-230

representation of the congruence upon the unit sphere, and the spherical representation of the congruence. When we put

$$\left(\frac{\partial X}{\partial u}\right)^2, \quad \mathcal{F} = \sum \frac{\partial X}{\partial u} \frac{\partial X}{\partial v}, \quad \mathcal{G} = \sum \left(\frac{\partial X}{\partial v}\right)^2,$$

the element of the spherical representation is

$$d\sigma^2 = \mathcal{E} du^2 + 2 \mathcal{F} du dv + \mathcal{G} dv^2$$

$$\frac{\partial X}{\partial u}, \quad f = \sum \frac{\partial x}{\partial v} \frac{\partial X}{\partial u}, \quad f' = \sum \frac{\partial x}{\partial u} \frac{\partial X}{\partial v}, \quad g = \sum \frac{\partial x}{\partial v} \frac{\partial X}{\partial v},$$

the second quadratic form

$$\sum dx dX = e du^2 + (f + f') du dv + g dv^2,$$

is fundamental in the theory of congruences.

Congruences. Ruled surfaces of a congruence. If a surface S' is normal to the congruence, the coordinates of a point on S' are given by

$$x + tX, \quad y' = y + tY, \quad z' = z + tZ,$$

where t is the distance from the surface of reference to S' . If S' is normal to the congruence, we must have

$$\sum X d(x + tX) = 0,$$

which is equivalent to

$$\sum X \frac{\partial x}{\partial u} + \frac{\partial t}{\partial u} = 0, \quad \sum X \frac{\partial x}{\partial v} + \frac{\partial t}{\partial v} = 0$$

These equations may be differentiated with respect to v and u respectively. If the resulting equations be subtracted, we obtain

$$f = f'$$

When this condition is satisfied, the function t given by (8) satisfies equation (7). Since t involves an arbitrary constant, equations (6) define a family of parallel surfaces to the congruence. Hence

a necessary and sufficient condition for a normal congruence is that the parallel surfaces be equal.

The congruence which pass through a curve on the surface S form a ruled surface. Such a curve, and

consequently a ruled surface of the congruence, is determined by a relation between u and v . Hence a differential equation of the

$$(10) \quad Mdu + Ndv = 0$$

defines a family of ruled surfaces of the congruence. We consider a line $l(u, v)$ of the congruence and the ruled surface Σ of the family upon which l is a generator, we say that Σ passes through l . We apply to Σ the results of §§ 103, 104.

If ds_0 denotes the linear element of the curve C in which Σ intersects the surface of reference, it follows from (VII, 54), (3), and (5) that the quantities a^2 and b for Σ have the values

$$(11) \quad \begin{cases} a^2 = \sum \left(\frac{dX}{ds_0} \right)^2 = \frac{d\sigma^2}{ds_0^2}, \\ b = \sum \frac{dX}{ds_0} \frac{dx}{ds_0} = \frac{e du^2 + (f + f') du dv + g dv^2}{ds_0^2} \end{cases}$$

From (VII, 58) we have that the direction-cosines λ, μ, ν common perpendicular to l and to the line l' of parameters $u + du, v + dv$, where dv/du is given by (10), have the values

$$(12) \quad \lambda = \left(Y \frac{dZ}{d\sigma} - Z \frac{dY}{d\sigma} \right), \quad \mu = \left(Z \frac{dX}{d\sigma} - X \frac{dZ}{d\sigma} \right), \quad \nu = \left(X \frac{dY}{d\sigma} - Y \frac{dX}{d\sigma} \right)$$

which, by means of (V, 31), are reducible to

$$(13) \quad \lambda = \frac{\left(\mathcal{E} \frac{\partial X}{\partial v} - \mathcal{F} \frac{\partial X}{\partial u} \right) du + \left(\mathcal{F} \frac{\partial X}{\partial v} - \mathcal{G} \frac{\partial X}{\partial u} \right) dv}{\sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2} d\sigma},$$

and similar expressions for μ and ν .

From (12) it follows that

$$\lambda \frac{dY}{d\sigma} + \mu \frac{dZ}{d\sigma} + \nu \frac{dX}{d\sigma} = 0$$

Since $dX/d\sigma, dY/d\sigma, dZ/d\sigma$ are the direction-cosines of the tangent to the spherical representation of the generators of Σ , we have the theorem

Given a ruled surface Σ of a congruence. Let C be the curve on the unit sphere which represents Σ , and M the point of C corresponding to a generator L of Σ , the limiting position of the common perpendicular to L and a near-by generator of Σ is perpendicular to the tangent to C at M .

62 Limit points Principal surfaces By means of (VII, 62) (12) we find that the expression for the shortest distance δ between l and l' is, to within terms of higher order,

$$\delta = \frac{ds_0^2}{d\sigma} \begin{vmatrix} \frac{dx}{ds_0} & \frac{dy}{ds_0} & \frac{dz}{ds_0} \\ X & Y & Z \\ \frac{dX}{ds_0} & \frac{dY}{ds_0} & \frac{dZ}{ds_0} \end{vmatrix} = \Sigma \lambda dx$$

When the values (13) for λ, μ, ν are substituted in the right-hand member of this equation, the result is reducible to

$$\delta = \frac{1}{\mathcal{H} d\sigma} \begin{vmatrix} \mathcal{E} du + \mathcal{F} dv, & \mathcal{F} du + \mathcal{G} dv \\ e du + f dv, & f' du + g dv \end{vmatrix}$$

If N denotes the point where this line of shortest distance meets l , the locus of N is the line of striction of Σ . Hence the distance of l from the surface S , measured along l , is given by (VII, 65), if we denote it by r , we have, from (11),

$$r = - \frac{e du^2 + (f + f') du dv + g dv^2}{\mathcal{E} du^2 + 2 \mathcal{F} du dv + \mathcal{G} dv^2}$$

For the present* we exclude the case where the coefficients of the two quadratic forms are proportional. Hence r varies with the value of dv/du , that is, with the ruled surface Σ through l . If we limit our consideration to real surfaces Σ , the denominator is always positive, and consequently the quantity r has a finite maximum and minimum. In order to find the surfaces Σ for which r has these limiting values, we replace dv/du by t , and obtain

$$r = - \frac{e + (f + f')t + gt^2}{\mathcal{E} + 2 \mathcal{F}t + \mathcal{G}t^2}$$

we equate to zero the derivative of the right-hand member with respect to t , we get

$$t) (\mathcal{E} + \mathcal{F}t) \left[\frac{1}{2}(f + f') + gt \right] - (\mathcal{F} + \mathcal{G}t) \left[e + \frac{1}{2}(f + f')t \right] = 0,$$

quadratic in t . Since $\mathcal{E}\mathcal{G} - \mathcal{F}^2 > 0$, we may apply to this equation reasoning similar to that used in connection with equation (IV, 21),

* Cf Ex 1, § 171

and thus prove that it has two real roots. The corresponding values of r follow from (16) when these values of t are substituted in the latter. Because of (17) the resulting equation may be written

$$\bar{r} = -\frac{e + \frac{1}{2}(f+f')\bar{t}}{\mathcal{E} + \mathcal{F}\bar{t}} = -\frac{\frac{1}{2}(f+f') + g\bar{t}}{\mathcal{F} + \mathcal{G}\bar{t}},$$

$\bar{r}$ indicates a root of (17) and $\bar{r}$ the corresponding value of r . In we write the preceding equations in the form

$$\begin{aligned} [\mathcal{E}\bar{r} + e] + [\mathcal{F}\bar{r} + \frac{1}{2}(f+f')] \bar{t} &= 0, \\ [\mathcal{F}\bar{r} + \frac{1}{2}(f+f')] + [\mathcal{G}\bar{r} + g] \bar{t} &= 0, \end{aligned}$$

eliminate $\bar{t}$, we obtain the following quadratic in $\bar{r}$

$$(18) \quad (\mathcal{E}\mathcal{G} - \mathcal{F}^2)\bar{r}^2 + [g\mathcal{E} - (f+f')\mathcal{F} + e\mathcal{G}]\bar{r} + eg - \left(\frac{f+f'}{2}\right)^2 = 0$$

If r_1 and r_2 denote the roots of this equation, we have

$$(19) \quad \begin{cases} r_1 + r_2 = \frac{(f+f')\mathcal{F} - g\mathcal{E} - e\mathcal{G}}{\mathcal{E}\mathcal{G} - \mathcal{F}^2}, \\ r_1 r_2 = \frac{4eg - (f+f')^2}{4(\mathcal{E}\mathcal{G} - \mathcal{F}^2)} \end{cases}$$

The points on l corresponding to these values of r are called its *limit points*. They are the boundaries of the segment of l upon which lie the feet of each perpendicular common to it and to a near-by line of the congruence. The ruled surfaces of the congruences which pass through l and are determined by equation (17) are called the *principal surfaces* for the line. There are two of them, and their tangent planes at the limit points are determined by l and by the perpendiculars of shortest distance at the limit points. They are called the *principal planes*.

In order to find other properties of the principal surfaces, we imagine that the parametric curves upon the sphere represent these surfaces. If equation (17) be written

$$(20) \quad \begin{vmatrix} \mathcal{E} du + \mathcal{F} dv, & \mathcal{G} du + \mathcal{H} dv \\ e du + \frac{1}{2}(f+f') dv, & \frac{1}{2}(f+f') du + g dv \end{vmatrix} = 0,$$

1 that a necessary and sufficient condition that the ruled $v = \text{const}$, $u = \text{const}$ be the principal surfaces, is

$$\frac{1}{2}(f+f')\mathcal{E}-e\mathcal{F}=0, \quad g\mathcal{F}-\frac{1}{2}(f+f')\mathcal{G}=0$$

It follows that since the coefficients of the two fundamental quadratic forms are not proportional, we must have

$$\mathcal{F}=0, \quad f+f'=0$$

From the first of these equations and the preceding theorem follows that

Principal surfaces of a congruence are represented on the xy plane in an orthogonal system, and the two principal planes for any point are perpendicular to one another

This particular parametric system equation (13) reduces to

$$\lambda = \frac{\mathcal{E} \frac{\partial X}{\partial v} du - \mathcal{G} \frac{\partial X}{\partial u} dv}{\sqrt{\mathcal{E}\mathcal{G}} d\sigma},$$

the direction-cosines λ_1, μ_1, ν_1 of the perpendicular whose foot is the point on l corresponding to $v = \text{const}$ have the values

$$\lambda_1 = \frac{1}{\sqrt{\mathcal{E}}} \frac{\partial X}{\partial v}, \quad \mu_1 = \frac{1}{\sqrt{\mathcal{G}}} \frac{\partial Y}{\partial v}, \quad \nu_1 = \frac{1}{\sqrt{\mathcal{G}}} \frac{\partial Z}{\partial v}$$

The angle ω between the lines with these direction-cosines and the line with direction-cosines λ, μ, ν is given by

$$\cos \omega = \sum \lambda \lambda_1 = \frac{\sqrt{\mathcal{E}} du}{\sqrt{\mathcal{E} du^2 + \mathcal{G} dv^2}}$$

Values of r_1 and r_2 are now

$$r_1 = -\frac{e}{\mathcal{E}}, \quad r_2 = -\frac{g}{\mathcal{G}},$$

with the aid of (23) equation (15) can be put in the form

$$r = r_1 \cos^2 \omega + r_2 \sin^2 \omega$$

Hamilton's equation We remark that it is independent of the choice of parameters

163. Developable surfaces of a congruence Focal surfaces In order that a ruled surface be developable, it is necessary and sufficient that the perpendicular distance between very near generators be of the second or higher order. From (14) it follows that the ruled surfaces of a congruence satisfying the condition

$$(25) \quad \begin{vmatrix} \mathcal{E} du + \mathcal{F} dv, & \mathcal{F} du + \mathcal{G} dv \\ e du + f dv, & f' du + g dv \end{vmatrix} = 0$$

are developable. Unlike equation (20), the values of dv/du satisfying this equation are not necessarily real. We have then the theorem

Of all the ruled surfaces of a congruence through a line of its curve there are developables, but they are not necessarily real.

The normals to a real surface afford an example of a congruence with real developables, for, the normals along a line of curvature form a developable surface (§ 51). Since f and f' are equal in this case, equations (20) and (25) are equivalent. And, conversely, the surfaces are equivalent only in this case. Hence

When a congruence is normal, and only then, the principal surfaces are developable.

When a ruled surface is developable its generators are tangent to a curve at the points where the lines of shortest distance meet them. Hence each line of a congruence is tangent to two curves in space, real or imaginary according to the character of the roots of equation (25). The points of contact are called the *focal points* for the line. By means of (25) we find that the values of r at these points are given by

$$\rho = - \frac{e du + f dv}{f du + g dv} - \frac{f' du + g dv}{f du + g dv}$$

If these equations be written in the form

$$\begin{aligned} (\mathcal{E}\rho + e) du + (\mathcal{F}\rho + f) dv &= 0, \\ (\mathcal{F}\rho + f') du + (\mathcal{G}\rho + g) dv &= 0, \end{aligned}$$

and if du, dv be eliminated, we have

$$(26) \quad (\mathcal{E}\mathcal{G} - \mathcal{F}^2)\rho^2 + [\mathcal{G}e - (f + f')\mathcal{F} + e\mathcal{F}]\rho + eg - ff' = 0$$

Let ρ_1, ρ_2 denote the roots of this equation, it follows that

$$\begin{cases} \rho_1 + \rho_2 = \frac{(f+f')\mathcal{F} - g\mathcal{E} - e\mathcal{G}}{\mathcal{E}\mathcal{G} - \mathcal{F}^2}, \\ \rho_1\rho_2 = \frac{eg - ff'}{\mathcal{E}\mathcal{G} - \mathcal{F}^2} \end{cases}$$

9) and (27) it is seen that

$$\begin{cases} r_1 + r_2 = \rho_1 + \rho_2, \\ (r_1 - r_2)^2 - (\rho_1 - \rho_2)^2 = \frac{(f-f')^2}{(\mathcal{E}\mathcal{G} - \mathcal{F}^2)} \end{cases}$$

Results may be interpreted as follows

Mid-points of the two segments bounded respectively by the limits and by the focal points coincide

Point is called the middle point of the line and its locus the surface of the congruence

Distance between the focal points is never greater than that the limit points. They coincide when the congruence is normal

Equation (24) may be written in the forms

$$\cos^2 \omega = \frac{r - r_2}{r_1 - r_2}, \quad \sin^2 \omega = \frac{r_1 - r}{r_1 - r_2}$$

If ω_1 and ω_2 denote the values of ω corresponding to the two developable surfaces, we have

$$\begin{aligned} \cos^2 \omega_1 &= \frac{\rho_1 - r_2}{r_1 - r_2}, & \sin^2 \omega_1 &= \frac{r_1 - \rho_1}{r_1 - r_2}, \\ \cos^2 \omega &= \frac{\rho - r}{r_1 - r_2}, & \sin^2 \omega &= \frac{r_1 - \rho}{r_1 - r_2} \end{aligned}$$

From these and the first of (28) it follows that

$$\cos^2 \omega_1 = \sin^2 \omega_2, \quad \sin^2 \omega_1 = \cos^2 \omega_2,$$

$$\cos 2\omega_1 + \cos 2\omega_2 = 0,$$

Consequently

$$\omega_1 + \omega_2 = \frac{\pi}{2} \pm n\pi,$$

$$\omega_1 - \omega_2 = \frac{\pi}{2} \pm n\pi,$$

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denotes any integer. If the latter equation be true, the developable surfaces are represented on the sphere by an orthogonal system, as follows from the theorem at the close of § 161. But the condition that equation (25) define an orthogonal system on the sphere is $f=f'$, that is, the congruence must be normal in this case the principal surfaces are the developables, equation (30) as well as (31) is satisfied. Hence equation (30) is the final solution of (29).

planes through l which make the angles ω_1, ω_2 with the focal plane $\omega = 0$ are called the *focal planes* for the line, the tangent planes to the two developable surfaces through l . Incidentally we have proved the theorem

A necessary and sufficient condition that the two focal planes for a line of a congruence be perpendicular is that the congruence be normal.

And from equation (30) it follows that

The focal planes are symmetrically placed with respect to the principal planes in such a way that the angles formed by the two pairs of planes have the same bisecting planes.

If θ denote the angle between the focal planes, then

$$\theta = \omega_2 - \omega_1 = \frac{\pi}{2} - 2\omega_1,$$

and

$$(32) \quad \sin \theta = \cos 2\omega_1 = \cos^2 \omega_1 - \cos^2 \omega = \frac{\rho_1 - \rho}{r_1 - r}$$

The loci of the focal points of a congruence are called its *focal surfaces*. Each line of the congruence touches both surfaces, being tangent to the edges of regression of the two developables through it. By reasoning similar to that employed in the discussion of surface of center (§ 74) we prove the theorem

A congruence may be regarded as two families of developable surfaces. Each focal surface is touched by the developables of one family along their edges of regression and enveloped by those of the other family along the curves conjugate to these edges.

The preceding theorem shows that of the two focal planes through a line l one is tangent to the focal surface S_1 and the other is the

ing plane of the edge of regression on S_1 to which l is tangent. Similar results hold for S_2 . When the congruence is normal, the planes are perpendicular, and consequently these edges of regression are geodesics on S_1 and S_2 . Since the converse is true (6), we have

Necessary and sufficient condition that the tangents to a family of curves on a surface form a normal congruence is that the curves be

EXAMPLES

Let X, Y, Z are the direction-cosines of the normal to a minimal surface at (x, y, z) , the line whose direction-cosines are $Y, -X, Z$ and which passes through the point $(x, y, 0)$ generates a normal congruence.

Prove that the tangent planes to two confocal quadrics at the points of common tangent are perpendicular, and consequently that the common tangents to two confocal quadrics form a normal congruence.

Find the congruence of common tangents to the paraboloids

$$x^2 + y^2 = 2az, \quad x^2 + y^2 = -2az,$$

and find the focal surfaces.

Two ruled surfaces through a line L are represented on the sphere by great circles, their lines of striction meet L at points equally distant from the line.

Prove that the focal planes for each line of a congruence meet under the same angle, it is necessary and sufficient that the osculating planes of the edges of the developables meet the tangent planes to the focal surfaces under the same angle.

Necessary and sufficient condition that a surface of reference of a congruence is a middle surface is $g\mathcal{C} - (f + f')\mathcal{F} + e\mathcal{G} = 0$.

Associate normal congruences If we put

$$\gamma = \sum_1 \frac{\partial u}{\partial v}, \quad \gamma_1 = \sum_1 X \frac{\partial x}{\partial v},$$

then (8) may be replaced by

$$t = c - \int \gamma du + \gamma_1 dv,$$

where c is a constant. Now equation (9) is equivalent to

$$\frac{\partial \gamma}{\partial v} = \frac{\partial \gamma_1}{\partial u}$$

In consequence of this condition equation (34) may be written

$$(36) \quad t = c - \phi(u_1),$$

where u_1 is a function of u and v thus defined. If the orthogonal trajectories of the curves $u_1 = \text{const}$ be taken as parametric curve $v_1 = \text{const}$, it follows from (36) and from equations in u_1 and v_1 analogous to (33) and (34) that

$$\sum X \frac{\partial x}{\partial v_1} = 0$$

From this result follows the theorem

The lines of a normal congruence cut orthogonally the curves of the surface of reference at whose points t is constant

If θ denotes the angle which a line of the congruence make with the normal to the surface of reference at the point of intersection, we have

$$(37) \quad \sin \theta = \sum \frac{X}{\sqrt{E}} \frac{\partial x}{\partial u_1} = \frac{\phi'(u_1)}{\sqrt{E}},$$

where the linear element of the surface is

$$ds^2 = E du_1^2 + G dv_1^2$$

If S be taken for the surface of reference of a second congruence whose direction-cosines X_1, Y_1, Z_1 satisfy the conditions

$$\sum \lambda_1 \frac{\partial x}{\partial u_1} = \phi_1'(u_1), \quad \sum \lambda_1 \frac{\partial x}{\partial v_1} = 0,$$

where $\phi_1(u_1)$ is any function whatever of u_1 , this congruence normal and t_1 has the value

$$t_1 = c_1 - \phi_1(u_1)$$

Since ϕ_1 is any function, there is a family of these normal congruences which we call the *associates* of the given congruence and of one another. Through any point of the surface of reference there passes a line of each congruence, and all of these lines lie in the plane normal to the curve $u_1 = \text{const}$ through the point. Hence

The two lines of two associate congruences through the same point of the surface of reference lie in a plane normal to the surface

joining with equation (37) a similar one for an associate congruence, we have

$$\frac{\sin \theta}{\sin \theta_1} = \frac{\phi'(u_1)}{\phi'_1(u_1)} = f(u_1)$$

we have the theorem

Ratio of the sines of the angles which the lines of two associate congruences make with the normal to their surface of reference is constant along the curves at whose points t is constant

In particular $f(u_1)$ in (38) is a constant, the former theorem and equation (38) constitute the laws of reflection and refraction of light, according as the constant is equal to or different from one. And so we have the theorem of Malus and Dupin

A bundle of rays of light forming a normal congruence be reflected or refracted any number of times by the surfaces of successive homogeneous media, the rays continue to constitute a normal congruence

Equations of (37) equation (36) can be put in the form

$$t = c - \int \sqrt{E} \sin \theta \, du_1$$

As a result follows the theorem of Beltrami *

If a surface of reference of a normal congruence be deformed in such a way that the directions of the lines of the congruence with respect to the surface be unaltered, the congruence continues to be normal

Derived congruences It is evident that the tangents to the lines of any one parameter family upon a surface S constitute a congruence. If these curves be taken for the parametric lines u , and then conjugates for $u = \text{const}$, the developables in this family have the curves $v = \text{const}$ for edges of regression, and the developables of the other family envelop S along the curves $u = \text{const}$. We may take S for the surface of reference. If S_1 be a focal surface, the lines of the congruence are tangent to S_1 as $u = \text{const}$ on S_1 . The tangents to the curves $v = \text{const}$ on S_1 form a second congruence of which S_1 is one focal surface, the second surface S' is uniquely determined. Moreover, the

* (Giornale di matematiche Vol II (1864) p 281)

lines of the second congruence are tangent to the curves $u = \text{const.}$ on S_2 . In turn we may construct a third congruence of tangents to the curves $v = \text{const.}$ on S_2 . This process may be continued indefinitely unless one of these focal surfaces reduces to a curve, or is infinitely distant.

In like manner we get a congruence by drawing tangents to the curves $u = \text{const.}$ on S , which is one focal surface, and the other, S_{-1} , is completely determined. The tangents to the curves $u = \text{const.}$ on S_{-1} form still another, and so on. In this way we obtain a suite of surfaces

$$S_{-2}, S_{-1}, S, S_1, S_2, \dots,$$

which is terminated only when a surface reduces to a curve, or its points are infinitely distant. Upon each of these surfaces the parametric curves form a conjugate system. The congruences thus obtained have been called *derived congruences* by Darboux*. It is clear that the problem of finding all the derived congruences of a given one reduces to the integration of the equation of its developables (25), for, when the developables are known we have the conjugate system on its focal surfaces.

In order to derive the analytical expressions for these results, we recall (§ 80) that the coordinates x, y, z of S are solutions of an equation of the form

$$(39) \quad \frac{\partial^2 \theta}{\partial u \partial v} + a \frac{\partial \theta}{\partial u} + b \frac{\partial \theta}{\partial v} = 0,$$

where a and b are determinate functions of u and v . If the coordinates of S_1 be denoted by x_1, y_1, z_1 , they are given by

$$x_1 = x + \lambda_1 \frac{\partial x}{\partial u}, \quad y_1 = y + \lambda_1 \frac{\partial y}{\partial u}, \quad z_1 = z + \lambda_1 \frac{\partial z}{\partial u},$$

where $\lambda_1 \sqrt{E}$ measures the distance between the focal points. But as the lines of the congruence are tangent to the curves $u = \text{const.}$ on S_1 , we must have

$$(40) \quad \frac{\partial x_1}{\partial v} = \mu_1 \frac{\partial x}{\partial v}, \quad \frac{\partial y_1}{\partial v} = \mu_1 \frac{\partial y}{\partial v}, \quad \frac{\partial z_1}{\partial v} = \mu_1 \frac{\partial z}{\partial v},$$

is a determinate function of u and v . When the above x_1 is substituted in the first of these equations, the result is, by means of (39), to

$$\left(\frac{\partial \lambda_1}{\partial v} - a\lambda_1 - \mu_1\right) \frac{\partial x}{\partial u} + (1 - b\lambda_1) \frac{\partial x}{\partial v} = 0$$

same equation is true for y and z , the quantities in parentheses be zero, that is,

$$\lambda_1 = \frac{1}{b}, \quad \mu_1 = \frac{\partial}{\partial v} \frac{1}{b} - \frac{a}{b}$$

surface S_1 is defined by

$$x_1 = x + \frac{1}{b} \frac{\partial x}{\partial u}, \quad y_1 = y + \frac{1}{b} \frac{\partial y}{\partial u}, \quad z_1 = z + \frac{1}{b} \frac{\partial z}{\partial u},$$

equations (40) become

$$= \left(\frac{\partial}{\partial v} \frac{1}{b} - \frac{a}{b}\right) \frac{\partial x}{\partial u}, \quad \frac{\partial y_1}{\partial v} = \left(\frac{\partial}{\partial v} \frac{1}{b} - \frac{a}{b}\right) \frac{\partial y}{\partial u}, \quad \frac{\partial z_1}{\partial v} = \left(\frac{\partial}{\partial v} \frac{1}{b} - \frac{a}{b}\right) \frac{\partial z}{\partial u}$$

likewise in a similar manner, we find that S_{-1} is defined by

$$x_{-1} = x + \frac{1}{a} \frac{\partial x}{\partial v}, \quad y_{-1} = y + \frac{1}{a} \frac{\partial y}{\partial v}, \quad z_{-1} = z + \frac{1}{a} \frac{\partial z}{\partial v},$$

$$\frac{\partial x_{-1}}{\partial u} = \left(\frac{\partial}{\partial u} \frac{1}{a} - \frac{b}{a}\right) \frac{\partial x}{\partial v},$$

expressions in y_{-1} and z_{-1}

(41) and (43) it is seen that the surface S_1 or S_{-1} is at infinity according as b or a is zero. When a and b are both zero, the surface is of translation (§ 81). Hence the tangents to the surfaces of a surface of translation form two congruences for which the other focal surface is at infinity.

For S_1 to be a curve, x_1, y_1, z_1 must be functions of u alone; it follows that the condition for this is

$$\frac{\partial}{\partial v} \frac{1}{b} = \frac{a}{b}$$

under the condition that S_{-1} be a curve is

$$\frac{\partial}{\partial u} \frac{1}{a} = \frac{b}{a}$$

The functions h and k , defined by

$$h = \frac{\partial a}{\partial u} + ab, \quad k = \frac{\partial b}{\partial v} + ab,$$

called the invariants of the differential equation (39) Hence the above results may be stated

A necessary and sufficient condition that the focal surface S_1 or S_2 be a curve is that the invariant k or h respectively of the point equation of S be zero

166 Fundamental equations of condition We have seen (§ 160) that with every congruence there are associated two quadratic differential forms. Now we shall investigate under what conditions two quadratic forms determine a congruence. We assume that we have two such forms and that there is a corresponding congruence whose tangents to the parametric curves on the surface of reference at a point are determined by the angles which they make with the tangents to the parametric curves of the spherical representation of the congruence at the corresponding point, and with the normal to the unit sphere. Hence we have the relations

$$\left\{ \begin{aligned} \frac{\partial x}{\partial u} &= \alpha \frac{\partial X}{\partial u} + \beta \frac{\partial Y}{\partial v} + \gamma X, \\ \frac{\partial x}{\partial v} &= \alpha_1 \frac{\partial X}{\partial u} + \beta_1 \frac{\partial Y}{\partial v} + \gamma_1 X, \end{aligned} \right.$$

and similar equations in y and z , where $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ are functions of u and v . If we multiply these equations by $\frac{\partial X}{\partial u}, \frac{\partial Y}{\partial u}, \frac{\partial Z}{\partial u}$ respectively, and add, also by $\frac{\partial X}{\partial v}, \frac{\partial Y}{\partial v}, \frac{\partial Z}{\partial v}$ and by X, Y, Z we obtain

$$\begin{aligned} e &= \alpha \epsilon + \beta \zeta, & f &= \alpha_1 \epsilon + \beta_1 \zeta, & \gamma_1 &= \sum \lambda \frac{\partial \epsilon}{\partial u}, \\ f' &= \alpha \zeta + \beta \epsilon, & g &= \alpha_1 \zeta + \beta_1 \epsilon, & \gamma_1 &= \sum \lambda \frac{\partial \epsilon}{\partial v}, \end{aligned}$$

from which we derive

$$\left\{ \begin{aligned} \alpha &= \frac{\epsilon \zeta - f' \zeta}{\epsilon \zeta - \zeta^2}, & \beta &= \frac{f' \epsilon - \epsilon \zeta}{\epsilon \zeta - \zeta^2}, & \gamma &= \sum \lambda \frac{\partial \epsilon}{\partial u}, \\ \alpha_1 &= \frac{f \zeta - g \zeta}{\epsilon \zeta - \zeta^2}, & \beta_1 &= \frac{g \epsilon - f \zeta}{\epsilon \zeta - \zeta^2}, & \gamma_1 &= \sum \lambda \frac{\partial \epsilon}{\partial v} \end{aligned} \right.$$

that equations (44) be consistent, we must have

$$\frac{\partial}{\partial u} \left(\frac{\partial x}{\partial v} \right) = \frac{\partial}{\partial v} \left(\frac{\partial x}{\partial u} \right),$$

consequence of equations (V, 22), is reducible to the form

$$R \frac{\partial X}{\partial u} + S \frac{\partial X}{\partial v} + TX = 0,$$

S , T are determinate functions. Since this equation must hold by Y and Z also, we must have $R = 0$, $S = 0$, $T = 0$. Values of α , β , α_1 , β_1 , from (45), are substituted in these we have

$$\frac{\partial f}{\partial u} - \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' e + \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}' f - \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' f' + \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' g + \mathcal{F}\gamma - \mathcal{G}\gamma_1 = 0,$$

$$\frac{\partial g}{\partial u} - \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' e + \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' f - \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}' f' + \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' g + \mathcal{G}\gamma - \mathcal{F}\gamma_1 = 0,$$

$$\frac{\partial \gamma_1}{\partial u} + f - f' = 0$$

ely, when we have a quadratic form whose curvature may be taken as the linear element of the spherical representation of a congruence, which is determined by any set of e , f , f' , g , γ , γ_1 satisfying equations (47). For, when conditions are satisfied so also is (46), and consequently coordinates of the surface of reference are given by the system (44).

Finally we remark that when the congruence is normal, and the surface of reference is one of the orthogonal surfaces, the last condition (47) is satisfied identically, and the first two reduce to Ricci's equations (V, 27).

Finally these results lead to the determination of the congruences which admit a spherical representation of their principal surfaces, with a given representation of their developables.

Spherical representation of principal surfaces and of developables. A necessary and sufficient condition that the principal surface of a congruence cut the surface of reference in the parameter space is given by (21)

If we require that the surface of reference be the middle surface of the congruence, and if r denote half the distance between the limit points, we have, from (15),

$$(48) \quad e = -r\mathcal{E}, \quad g = r\mathcal{G}$$

When these values are substituted in (47), the first two become

$$(49) \quad \begin{cases} \gamma_1 = -\frac{1}{\mathcal{E}} \frac{\partial}{\partial v} (r\mathcal{E}) - \sqrt{\frac{\mathcal{G}}{\mathcal{E}}} \frac{\partial}{\partial u} \left(\frac{f}{\sqrt{\mathcal{E}\mathcal{G}}} \right), \\ \gamma = \frac{1}{\mathcal{G}} \frac{\partial}{\partial u} (r\mathcal{G}) + \sqrt{\frac{\mathcal{E}}{\mathcal{G}}} \frac{\partial}{\partial v} \left(\frac{f}{\sqrt{\mathcal{E}\mathcal{G}}} \right), \end{cases}$$

and the last is reducible to *

$$(50) \quad 2 \frac{\partial^2 r}{\partial u \partial v} + \frac{\partial \log \mathcal{E}}{\partial v} \frac{\partial r}{\partial u} + \frac{\partial \log \mathcal{G}}{\partial u} \frac{\partial r}{\partial v} + \frac{\partial^2 \log \mathcal{E}\mathcal{G}}{\partial u \partial v} r \\ + \frac{\partial}{\partial u} \left[\sqrt{\frac{\mathcal{G}}{\mathcal{E}}} \frac{\partial}{\partial u} \left(\frac{f}{\sqrt{\mathcal{E}\mathcal{G}}} \right) \right] + \frac{\partial}{\partial v} \left[\sqrt{\frac{\mathcal{E}}{\mathcal{G}}} \frac{\partial}{\partial v} \left(\frac{f}{\sqrt{\mathcal{E}\mathcal{G}}} \right) \right] + 2f = 0$$

Moreover, equations (44) become

$$(51) \quad \frac{\partial x}{\partial u} = -r \frac{\partial X}{\partial u} - \frac{f}{\mathcal{G}} \frac{\partial X}{\partial v} + \gamma X, \quad \frac{\partial x}{\partial v} = \frac{f}{\mathcal{E}} \frac{\partial X}{\partial u} + r \frac{\partial X}{\partial v} + \gamma_1 X,$$

where γ and γ_1 are given by (49), and similar equations in y and z . Our problem reduces, therefore, to the determination of pairs of functions r and f which satisfy (50). Evidently either of these functions may be chosen arbitrarily and the other is found by the solution of a partial differential equation of the second order. Hence any orthogonal system on the unit sphere serves for the representation of the principal surfaces of a family of developables whose equations involve three arbitrary functions.

In order that the parametric curves on the sphere represent the developables of a congruence, it is necessary and sufficient that

$$f'_{\mathcal{E}} - e_{\mathcal{G}} = 0, \quad g_{\mathcal{G}} - f'_{\mathcal{E}} = 0,$$

as is seen from (25). If the surface of reference be the middle surface, and ρ denotes half the distance between the focal points, follows from (15) that

$$\rho = -\frac{e}{\mathcal{E}} = \frac{g}{\mathcal{G}}$$

g these equations with the above, we have

$$e = -\rho \mathcal{E}, \quad f = -f' = \rho \mathcal{F}, \quad g = \rho \mathcal{G}$$

se values are substituted in the first two of equations (47) resulting equations are solved for γ and γ_1 , we find

$$\gamma = \frac{\partial \rho}{\partial u} + 2 \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \rho, \quad \gamma_1 = -\frac{\partial \rho}{\partial v} - 2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \rho,$$

st of equations (47) reduces to

$$+ \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \frac{\partial \rho}{\partial u} + \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \frac{\partial \rho}{\partial v} + \left[\frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' + \frac{\partial}{\partial v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' + \mathcal{F} \right] \rho = 0$$

tion of this equation determines a congruence with the representation of its developables,* and the middle surface γ the quadratures

$$\begin{cases} \frac{\partial x}{\partial u} = -\rho \frac{\partial X}{\partial u} + \left(\frac{\partial \rho}{\partial u} + 2 \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \rho \right) X, \\ \frac{\partial x}{\partial v} = \rho \frac{\partial X}{\partial v} - \left(\frac{\partial \rho}{\partial v} + 2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \rho \right) X, \end{cases}$$

r expressions in y and z

he values (52) are substituted in (18) the latter becomes

$$(\rho^2 - \rho'^2) \tilde{r}^2 - \rho^2 \mathcal{E} \mathcal{G} = 0$$

itly equation (32) reduces to

$$\sin \theta = \frac{2\rho}{2r} = \frac{\sqrt{\rho^2 - \rho'^2}}{\sqrt{\rho^2 + \rho'^2}}$$

to equation (III, 16), we have

de between the focal planes of a congruence is equal to the area of the lines on the sphere representing the corresponding

ult is obtained readily from geometrical considerations
ndamental quantities for the focal surfaces We shall
of these results in deriving the expressions for the funda-
ntities of the focal surfaces S_1 and S_2 , which are defined by

$$\begin{aligned} x_1 &= x + \rho X, & y_1 &= y + \rho Y, & z_1 &= z + \rho Z, \\ x_2 &= x - \rho X, & y_2 &= y - \rho Y, & z_2 &= z - \rho Z \end{aligned}$$

It is due to Gukhard *Annales de l'Ecole Normale* Ser. 3, Vol. VI (1889),

From these and (54) we get

$$(55) \quad \begin{cases} \frac{\partial x_1}{\partial u} = 2 \left(\frac{\partial \rho}{\partial u} + \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \rho \right) X, & \frac{\partial x_1}{\partial v} = 2 \rho \left(\frac{\partial X}{\partial v} - \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' X \right), \\ \frac{\partial x_2}{\partial u} = -2 \rho \left(\frac{\partial X}{\partial u} - \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' X \right), & \frac{\partial x_2}{\partial v} = -2 \left(\frac{\partial \rho}{\partial v} + \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \rho \right) X \end{cases}$$

The coefficients of the linear elements of S_1 and S_2 , as derived from these formulas, are

$$(56) \quad \begin{cases} E_1 = 4 \left(\frac{\partial \rho}{\partial u} + \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \rho \right)^2, & F_1 = -4 \rho \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \left(\frac{\partial \rho}{\partial u} + \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \rho \right) \\ G_1 = 4 \rho^2 \left(\mathcal{S} + \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \right)^2, & H_1 = 4 \rho \sqrt{\mathcal{S}} \left(\frac{\partial \rho}{\partial u} + \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \rho \right), \end{cases}$$

and

$$(57) \quad \begin{cases} E_2 = 4 \rho^2 \left(\mathcal{C} + \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \right)^2, & F_2 = -4 \rho \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \left(\frac{\partial \rho}{\partial v} + \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \rho \right), \\ G_2 = 4 \left(\frac{\partial \rho}{\partial v} + \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \rho \right)^2, & H_2 = 4 \rho \sqrt{\mathcal{C}} \left(\frac{\partial \rho}{\partial v} + \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \rho \right) \end{cases}$$

The direction-cosines of the normals to S_1 and S_2 denoted $X_1, Y_1, Z_1, X_2, Y_2, Z_2$ respectively are found from the above equations and (V, 31) to have the values

$$X_1 = \frac{1}{H_1} \frac{\partial(y_1, z_1)}{\partial(u, v)} = \frac{1}{\sqrt{\mathcal{S}}} \frac{\frac{\partial Z}{\partial v} - \rho \frac{\partial Y}{\partial v}}{\frac{\partial \rho}{\partial u} + \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \rho} = \frac{1}{\sqrt{\mathcal{S}}} \left(\rho \frac{\partial Y}{\partial v} - \frac{\partial Z}{\partial v} \right),$$

$$X_2 = \frac{1}{H_2} \frac{\partial(y_2, z)}{\partial(u, v)} = \frac{1}{\sqrt{\mathcal{C}}} \frac{\frac{\partial Z}{\partial u} - \rho \frac{\partial Y}{\partial u}}{\frac{\partial \rho}{\partial v} + \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \rho} = \frac{1}{\sqrt{\mathcal{C}}} \left(\rho \frac{\partial Y}{\partial u} - \frac{\partial Z}{\partial u} \right),$$

and similar expressions for Y_1 and Z_1 . If these equations be differentiated, and the resulting equations be reduced by means of (V, 2) they can be put in the form

$$\frac{\partial X_1}{\partial u} = \frac{\mathcal{H}}{\sqrt{\mathcal{S}}} \left[\frac{1}{\mathcal{S}} \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \frac{\partial Y}{\partial v} + \Lambda \right], \quad \frac{\partial X_1}{\partial v} = \frac{\mathcal{H}}{\sqrt{\mathcal{S}}} \left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\}' \frac{Y}{\rho},$$

$$\frac{\partial X_2}{\partial u} = \frac{\mathcal{H}}{\sqrt{\mathcal{C}}} \left\{ \begin{smallmatrix} 11 \\ 2 \end{smallmatrix} \right\}' \frac{\partial X}{\partial u}, \quad \frac{\partial X_2}{\partial v} = \frac{\mathcal{H}}{\sqrt{\mathcal{C}}} \left[\frac{1}{\mathcal{C}} \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \frac{Y}{\rho} + \Lambda \right]$$

these expressions and (55) we obtain

$$\begin{cases} D_1 = -\sum \frac{\partial x_1}{\partial u} \frac{\partial X_1}{\partial u} = -\frac{2}{\sqrt{\mathcal{E}}} \left(\frac{\partial \rho}{\partial u} + \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \rho \right), \\ D_1' = -\sum \frac{\partial x_1}{\partial v} \frac{\partial X_1}{\partial u} = -\sum \frac{\partial x_1}{\partial u} \frac{\partial X_1}{\partial v} = 0, \\ D_1'' = -\sum \frac{\partial x_1}{\partial v} \frac{\partial X_1}{\partial v} = -\frac{2}{\sqrt{\mathcal{E}}} \left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\}' \rho, \end{cases}$$

$$D_2 = \frac{2}{\sqrt{\mathcal{E}}} \left\{ \begin{smallmatrix} 11 \\ 2 \end{smallmatrix} \right\}' \rho, \quad D_2' = 0, \quad D_2'' = \frac{2}{\sqrt{\mathcal{E}}} \left(\frac{\partial \rho}{\partial v} + \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \rho \right)$$

the foregoing formulas we derive the following expressions total curvature of S_1 and of S_2

$$K_1 = \frac{\mathcal{H}^2 \left\{ \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \right\}'}{4 \rho \mathcal{E}^2 \left(\frac{\partial \rho}{\partial u} + \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \rho \right)}, \quad K_2 = \frac{\mathcal{H}^2 \left\{ \begin{smallmatrix} 11 \\ 2 \end{smallmatrix} \right\}'}{4 \rho \mathcal{E}^2 \left(\frac{\partial \rho}{\partial v} + \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \rho \right)}$$

EXAMPLES

upon a surface of reference S of a normal congruence the curves orthogonals to the lines of the congruence are defined by $\phi(u, v) = \text{const}$, and θ denotes the angle between a line of the congruence and the normal to the surface at the meeting, then $\sin^2 \theta = \Delta_1 I(q)$ where the differential parameter is formed with respect to the linear element of S . Show that θ is constant along a line $\phi = \text{const}$ on the latter is a geodesic parallel.

then in the point equation of a surface, namely

$$a \frac{\partial^2 \theta}{\partial u \partial v} + b \frac{\partial^2 \theta}{\partial u^2} + c \frac{\partial^2 \theta}{\partial v^2} = 0,$$

zero the coordinates of the surface can be found by quadratures

and the derived congruences of the tangents to the parametric curves on a focal surface (Ex. 2 p. 267) and determine under what conditions the surface S_1 is a curve

and the equation of the type (39) which admits as solutions the quantities ρ given by (41)

then a congruence consists of the tangents to the lines of curvature in one on a surface the focal distances are equal to the radii of geodesic curvature of the lines of curvature in the other system

6 Let S be a surface referred to its lines of curvature, let s_1 and s_2 denote the s of the curves $v = \text{const.}$ and $u = \text{const.}$ respectively, r_1 and r_2 their radii of curvature, and R_1 and R_2 their radii of geodesic curvature, for the second focal sheet Σ_1 of the congruence of tangents to the curves $v = \text{const.}$ the linear element is reducible to

$$d\sigma_1^2 = (dR_2 + ds_1)^2 + \frac{R_2^2}{r_1^2} ds_1^2,$$

hence the curves $s_1 = \text{const.}$ are geodesics.

7 Show that Σ_1 of Ex 6 is developable when $r_1 = f(s_1)$, and determine the most general form of r_1 so that Σ_1 shall be developable

8 Determine the condition which ρ must satisfy in order that the asymptotic lines on either focal surface of a congruence shall correspond to a conjugate system on the other, and show that in this case

$$K_1 K_2 = - \left(\frac{\sin \theta}{2\rho} \right)^4,$$

where θ denotes the angle between the focal planes

9 In order that the focal surfaces degenerate into curves, it is necessary and sufficient that the spherical representation satisfy the conditions

$$\frac{\partial}{\partial u} \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}' = \frac{\partial}{\partial v} \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}' = \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}' \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}' - \mathcal{F}$$

10 Show that the surfaces orthogonal to a normal congruence of the type of Ex 9 are cycloids of Dupin

11 A necessary and sufficient condition that the second sheet of the congruence of tangents to a family of curves on a surface S be developable is that the curves be plane

169 Isotropic congruences An *isotropic congruence* is one whose focal surfaces are developables with minimal edges of regression. In § 31 we saw that $H = 0$ is a necessary and sufficient condition that a surface be of this kind. Referring to (56) and (57), we see that we must have

$$\rho \sqrt{\mathcal{E}} \left(\frac{\partial \rho}{\partial u} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}' \rho \right) = 0, \quad \rho \sqrt{\mathcal{E}} \left(\frac{\partial \rho}{\partial v} + \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}' \rho \right) = 0$$

From (54) it is seen that if ρ were zero the middle surface would be a point, and from (55) that if the expressions in parentheses were zero the surfaces S_1 and S would be curves. Consequently

$$(61) \quad \mathcal{E} = \mathcal{F} = 0$$

Conversely, if this condition be satisfied, S_1 and S are isotropic developables. Hence an isotropic congruence is one whose developables are represented on the sphere by minimal lines

sequence of (61) we have, from (52),

$$e = g = 0,$$

$f + f'$ also is zero, it follows that

$$dx dX + dy dY + dz dZ = 0$$

r is zero, so that all the lines of striction lie on the surface. Since (61) is a consequence of (62), we have the theorem of Ribaucour,* which is sometimes taken as the definition of isotropic congruences

Lines of striction of an isotropic congruence lie on the middle surface, and, conversely, when all the lines of striction lie on the middle surface, the congruence is isotropic, moreover, the middle surface corresponds to the spherical representation with orthogonality of parametric lines

We have also established the following theorem †

The middle surface of an isotropic congruence is a minimal surface

Since the minimal lines on the sphere are parametric, in order to prove this theorem it is only necessary to show that on the middle surface, that is, the envelope of the middle planes, the parametric lines form a conjugate system. If W denotes the distance of the middle plane from the origin, the condition is necessary and sufficient that the parametric lines be conjugate satisfy the equation

$$\frac{\partial \theta}{\partial u \partial v} + \rho \theta = 0$$

or

$$H = \sum \lambda x,$$

the aid of (A 22) we find

$$\frac{\partial H}{\partial u \partial v} = \frac{\partial \rho}{\partial u \partial v} + \rho \rho = H \rho$$

Equation (53) reduces to $\frac{\partial \rho}{\partial u \partial v} + \rho \rho = 0$, the function H is arbitrary.

* *Flaschenden ou Surfaces à Courbure Moyenne Nulle. Memoires Couronnés de Belgique. Vol. XLIV (1881) p. 63*
 † *Ibid.* p. 31

170 Congruences of Guichard. Guichard* proposed and solved the problem

To determine the congruences whose focal surfaces are met by the developables in the lines of curvature

With Bianchi we call them *congruences of Guichard*

We remark that a necessary and sufficient condition that a congruence be of this kind is that F_1 and F_2 of § 168 be zero. From (56) and (57) it is seen that this is equivalent to

$$(64) \quad \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' = \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' = 0$$

Comparing this result with § 78, we have the theorem

A necessary and sufficient condition that the developables of a congruence meet the focal surfaces in their lines of curvature is that the congruence be represented on the sphere by curves representing also the asymptotic lines on a pseudospherical surface

In this case the parameters can be so chosen that †

$$\mathcal{E} = \mathcal{F} = 1, \quad \mathcal{G} = -\cos \omega,$$

where ω is a solution of

$$\frac{\partial^2 \omega}{\partial u \partial v} = \sin \omega$$

In this case equation (53) is

$$(65) \quad \frac{\partial^2 \rho}{\partial u \partial v} = \rho \cos \omega$$

In particular, this equation is satisfied by 1, 1, $\angle(V, 22)$. If we replace ρ by X in (54), we have

$$\frac{\partial x}{\partial u} = 0, \quad \frac{\partial x}{\partial v} = 0$$

consequently, for the congruence determined by this value of ρ , the middle surface is a plane

From (55) it follows that the lines of the congruence are tangent to the lines of curvature $v = \text{const}$ on S_1 . Consequently they are

* *L. C.*, p. 346

† This is the only real solution of (64)

, the normals to one of the sheets of the evolute of S_1 call it Σ_1 . Hence the conjugate system on Σ_1 corre- to the lines of curvature on S_1 is represented on the the same lines as the developables of the congruence to (VI, 38), we see that condition (64) is equivalent to

$$\begin{Bmatrix} 11 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} = 0,$$

Christoffel symbols are formed with respect to the linear f Σ_1 . But these are the conditions that the parametric Σ_1 be geodesics (cf § 85). Surfaces with a conjugate geodesics were studied by Voss,* and on this account *surfaces of Voss*. Since the converse of the above results , have the following theorem of Guichard

nary and sufficient condition that the tangents to the lines are in one family of a surface form a congruence of is that one sheet of the evolute of the surface be a sur- face, and that the tangents constituting the congruence be h are parallel to the normals to the latter

notes the distance from the origin to the tangent plane face of Voss Σ_1 , then H_1 is a solution of equation (65)

Hence $H_1 + \kappa \rho$ is a solution of this equation, provided κ ant. But since the tangent plane to Σ_1 passes through ponding point of S_1 the above result shows that a plane the lines of the congruence, and which divides in con- the segment between the focal points, envelopes a sur- face. In particular we have the corollary

The envelop of a congruence of Guichard is a surface of Voss

eudospherical congruences. The lines joining correspond- on a pseudospherical surface S and on one of its Buklund S_1 (cf § 120) constitute an congruence. We

the distance between corresponding points is constant, the tangent planes to the two surfaces at these points r constant angle. From (32) it follows that the distance n limit points also is constant.

* *Monatsh. Math. Phys.* Vol. XVIII (1888) pp. 95-102

Conversely, when the angle between the focal planes of a congruence is constant, and consequently also the angle θ between the parametric lines on the sphere representing the developables, we have, from (V, 4),

$$(66) \quad \frac{1}{\mathcal{E}} \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' + \frac{1}{\mathcal{F}} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' = 0, \quad \frac{1}{\mathcal{E}} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' + \frac{1}{\mathcal{F}} \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' = 0$$

Furthermore, if the distance between the focal points is constant, we have $\rho = a$, and by (60)

$$K_1 = K_2 = -\frac{\sin^2 \theta}{4a^2}$$

Hence the two focal surfaces have the same constant curvature. Congruences of this kind were first studied by Bianchi*. He called them *pseudospherical congruences*.

In order that the two focal surfaces of the congruence be Backlund transforms of one another, it is necessary that their lines of curvature correspond. It is readily found that for both surfaces the equation of these lines is reducible by means of (66) to

$$\begin{aligned} \mathcal{E} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' du^2 - \left[\mathcal{E}\mathcal{F} + \mathcal{E} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}'^2 + \mathcal{F} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}'^2 \right] dudv \\ + \mathcal{F} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' dv^2 = 0 \end{aligned}$$

Moreover, the differential equation of the asymptotic lines on each surface is $\mathcal{E}du^2 - \mathcal{F}dv^2 = 0$. Hence we have the theorem:

On the focal surfaces of a pseudospherical congruence the lines of curvature correspond, and likewise the asymptotic lines.

The focal surfaces of a pseudospherical congruence are Backlund transforms of one another.

EXAMPLES

1. When the parameters of a congruence are any whatsoever, and likewise the surface of reference, a condition necessary and sufficient that a congruence be isotropic is

$$\begin{array}{ccc} c & f + f & g \\ \mathcal{E} & 2\mathcal{F} & \mathcal{F} \end{array}$$

2. A necessary and sufficient condition that a congruence be isotropic is that the locus of two points on each line at an equal constant distance from the middle surface shall describe applicable surfaces.

3. Show that equation (65) admits $\frac{\partial \omega}{\partial u}$ and $\frac{\partial \omega}{\partial v}$ as solutions. Prove that in each case one of the focal surfaces is a sphere.

* *Annali*, Ser. 2, Vol. XV (1887) pp. 161-172; also *Lezioni*, Vol. I pp. 323, 324.

mine all the congruences of Guichard for which one of the focal surfaces

a surface is referred to its lines of curvature, a necessary and sufficient condition that the tangents to the curves $v = \text{const.}$ shall form a congruence is

$$\frac{\partial}{\partial u} \left(\frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \right) = 0$$

mine the surfaces which are such that the tangents to the lines of each system form a congruence of Guichard

W-congruences We have just seen that the asymptotic lines of focal surfaces of a pseudospherical congruence correspond, as is true in the case of the congruences of normals to a surface (cf § 124). For this reason all congruences possessing this property are called *W-congruences*. We shall derive other properties of these congruences.

Condition that asymptotic lines correspond, namely

$$D_1 D_2'' = D_1'' D_2,$$

following form in consequence of (58) and (59)

$$\left(\frac{\partial \rho}{\partial u} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \rho \right) \left(\frac{\partial \rho}{\partial v} + \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \rho \right) = \rho^2 \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}'$$

From (60) it follows that a necessary and sufficient condition for a congruence to be a W-congruence is

$$K_1 K_2 = \left(\frac{\sin \theta}{2\rho} \right)^4$$

In order to obtain an idea of the analytical problem involved in the determination of W-congruences we suppose that we have two surfaces referred to their asymptotic lines, and inquire under what conditions the lines joining corresponding points on the surfaces are tangents to them. We assume that the coordinates of the surfaces are determined by means of the Beltrami formulas (cf § 79), thus

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} = \begin{pmatrix} \nu_1 & \nu_2 & \nu_3 \\ \nu_1' & \nu_2' & \nu_3' \\ \nu_1'' & \nu_2'' & \nu_3'' \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} = - \begin{pmatrix} \nu_1 & \nu_2 & \nu_3 \\ \nu_1' & \nu_2' & \nu_3' \\ \nu_1'' & \nu_2'' & \nu_3'' \end{pmatrix},$$

* Cf. Guichard, *Comptes Rendus*, Vol. CX (1890) pp. 126-127.

and similar equations in $\bar{y}$, $\bar{z}$, y , and z . The functions $\nu_1, \nu_2, \nu_3, \bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3$ respectively are solutions of equations of the form

$$(69) \quad \frac{\partial^2 \theta}{\partial u \partial v} = \lambda \theta, \quad \frac{\partial^2 \bar{\theta}}{\partial u \partial v} = \bar{\lambda} \bar{\theta},$$

and they are such that

$$(70) \quad \nu_1^2 + \nu_2^2 + \nu_3^2 = \alpha, \quad \bar{\nu}_1^2 + \bar{\nu}_2^2 + \bar{\nu}_3^2 = \bar{\alpha},$$

where α and $\bar{\alpha}$ are defined by

$$(71) \quad K = -\frac{1}{\alpha^2}, \quad \bar{K} = -\frac{1}{\bar{\alpha}^2}$$

Since ν_1, ν_2, ν_3 and $\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3$ are proportional to the direction-cosines of the normals to S and $\bar{S}$, the condition that the lines joining corresponding points be tangent to the surfaces S and $\bar{S}$ is

$$\nu_1(\bar{x} - x) + \nu_2(\bar{y} - y) + \nu_3(\bar{z} - z) = 0,$$

$$\bar{\nu}_1(\bar{x} - x) + \bar{\nu}_2(\bar{y} - y) + \bar{\nu}_3(\bar{z} - z) = 0$$

Hence

$$\frac{\bar{x} - x}{\nu_2 \bar{\nu}_3 - \nu_3 \bar{\nu}_2} = \frac{\bar{y} - y}{\nu_3 \bar{\nu}_1 - \nu_1 \bar{\nu}_3} = \frac{\bar{z} - z}{\nu_1 \bar{\nu}_2 - \nu_2 \bar{\nu}_1} = m,$$

where m denotes a factor of proportionality. In order to find its value, we notice that from these equations follow the relations

$$\begin{aligned} (2\rho)^2 &= \Sigma(\bar{x} - x)^2 = m^2 \Sigma(\nu_2 \nu_3 - \nu_1 \nu_1)^2 \\ &= m^2 \Sigma[\alpha \alpha - (\Sigma \nu_1 \nu_1)^2] = m \alpha \bar{\alpha} \sin^2 \theta \end{aligned}$$

where θ denotes the angle between the focal planes. If this value of 2ρ and the values of K and $\bar{K}$ from (71) be substituted in (67), it is found that $m^4 = 1$. We take $m = 1$, thus fixing the signs of $\nu_1, \nu_2, \bar{\nu}_3$, and the above equations become

$$(72) \quad \bar{x} - x = \nu_2 \nu_3 - \nu_3 \nu_1, \quad \bar{y} - y = \nu_1 \nu_1 - \nu_1 \nu_2, \quad \bar{z} - z = \nu_1 \nu_2 - \nu_2 \nu_1$$

If the first of these equations be differentiated with respect to u , the result is reducible by (68) to

$$(\nu_2 - \nu_3) \left(\frac{\partial \bar{\nu}}{\partial u} + \frac{c \nu}{c u} \right) = (\nu_2 - \nu_1) \left(\frac{c \nu_1}{c u} + \frac{c \nu_2}{c u} \right)$$

g in like manner with the others, and also differentiating not to v , we are brought to

$$\begin{cases} \frac{\partial}{\partial u}(\bar{v}_i + v_i) = k(\bar{v}_i - v_i), \\ \frac{\partial}{\partial v}(\bar{v}_i - v_i) = l(\bar{v}_i + v_i), \end{cases} \quad (i=1, 2, 3)$$

and k are factors of proportionality to be determined. First of these equations be differentiated with respect to v , in reduction we make use of the second and of (69), we find

$$\left(\bar{\lambda} - kl - \frac{\partial k}{\partial v}\right)\bar{v}_i + \left(\lambda - kl + \frac{\partial k}{\partial v}\right)v_i = 0$$

Further, if the second of the above equations be differentiated with respect to u , we obtain

$$\left(\lambda - kl - \frac{\partial l}{\partial u}\right)\bar{v}_i - \left(\lambda - kl + \frac{\partial l}{\partial v}\right)v_i = 0$$

These equations are true for $i=1, 2, 3$, the quantities in brackets must be zero. This gives

$$\lambda = \frac{\partial k}{\partial u} + kl, \quad \lambda = -\frac{\partial k}{\partial v} + kl, \quad \frac{\partial l}{\partial u} = \frac{\partial k}{\partial v}$$

Once with the last we put

$$k = \frac{\partial}{\partial u} \log \frac{1}{\theta_1}, \quad l = \frac{\partial}{\partial v} \log \frac{1}{\theta_1},$$

these become

$$\lambda = \theta_1 \frac{\partial^2}{\partial u \partial v} \left(\frac{1}{\theta_1} \right), \quad \lambda = \frac{1}{\theta_1} \frac{\partial^2 \theta_1}{\partial u \partial v}$$

Equations (69) may be written

$$\frac{\partial^2 \theta}{\partial u \partial v} = \frac{1}{\theta_1} \frac{\partial \theta_1}{\partial u} \frac{\partial \theta}{\partial v} - \frac{\partial \theta}{\partial u} \frac{\partial^2 \theta_1}{\partial u \partial v} \left(\frac{1}{\theta_1} \right) \theta,$$

hence it follows that θ_1 is a solution of the first of equations (73) and $1/\theta_1$ of the second. Moreover, equations (73) may be written in the form

$$\frac{\partial}{\partial u} \left(\theta_1 v_i \right) = \frac{\partial \theta_1}{\partial u} v_i + \theta_1 \frac{\partial v_i}{\partial u}, \quad \frac{\partial}{\partial v} \left(\theta_1 v_i \right) = \frac{\partial \theta_1}{\partial v} v_i + \theta_1 \frac{\partial v_i}{\partial v}$$

Hence if θ_1 be a known solution of the first of equations (69), we obtain by quadratures three functions $\bar{v}_i$, which lead by the quadratures (68) to a surface $\bar{S}$. The latter is referred to its asymptotic lines and the joins of corresponding points on S and $\bar{S}$ are tangent to the latter. And so we have

If a surface S be referred to its asymptotic lines, and the equations of the surface be in the Lelievre form, each solution of the corresponding equation

$$\frac{\partial^2 \theta}{\partial u \partial v} = \lambda \theta$$

determines a surface $\bar{S}$, found by quadratures, such that S and $\bar{S}$ are the focal surfaces of a W -congruence

Comparing (74) with (XI, 13), we see that if we put

$$x_1 = \theta_1 \bar{v}_1, \quad y_1 = \theta_1 \bar{v}_2, \quad z_1 = \theta_1 \bar{v}_3,$$

the locus of the point (x_1, y_1, z_1) corresponds to S with orthogonality of linear elements. Hence $\bar{v}_1, \bar{v}_2, \bar{v}_3$ are proportional to the direction-cosines of the generatrices of an infinitesimal deformation of S , so that we have

Each focal surface of a W -congruence admits of an infinitesimal deformation whose generatrices are parallel to the normals to the other focal surface

Since the steps in the preceding argument are reversible, we have the theorem

The tangents to a surface which are perpendicular to the generatrices of an infinitesimal deformation of the latter constitute a W -congruence of the most general kind and the normals to the other surface are parallel to the generatrices of the deformation

173 Congruences of Ribaucour In his study of surfaces corresponding with orthogonality of linear elements Ribaucour considered the congruence formed by the lines through points on one surface parallel to the normals to a surface corresponding with the former in this manner. Bianchi * calls such a congruence a *congruence of Ribaucour*, and the second surface the *director surface*.

In order to ascertain the properties of such a congruence, we recall the results of § 153. Let S_1 be taken for the surface of

and draw lines parallel to the normals to S . If the latter to its asymptotic lines, it follows from (XI, 6) that

$$\sum \frac{\partial x_1}{\partial u} \frac{\partial \lambda}{\partial u} = -\frac{D' \mathcal{E} \phi}{HK}, \quad f = \sum \frac{\partial x_1}{\partial v} \frac{\partial X}{\partial u} = \frac{D' \mathcal{S} \phi}{HK},$$

$$\sum \frac{\partial x_1}{\partial u} \frac{\partial \lambda}{\partial v} = -\frac{D' \mathcal{S} \phi}{HK}, \quad g = \sum \frac{\partial x_1}{\partial v} \frac{\partial X}{\partial v} = \frac{D' \mathcal{E} \phi}{HK}$$

values satisfy the conditions

$$f' \mathcal{E} - e \mathcal{S} = 0, \quad g \mathcal{S} - f \mathcal{E} = 0,$$

surfaces $u = \text{const}$, $v = \text{const}$ are the developables. And $\rho_1 + \rho_2$ is equal to zero, S_1 is the middle surface of the . . . But the parametric curves on S_1 form a conjugate system in the asymptotic lines on S are parametric. Hence we theorem

Developable surfaces of a congruence of Ribaucour cut the face in a conjugate system

I* proved that this property is characteristic of congruences of Ribaucour. In order to obtain this result, we differentiate equations (54) with respect to v , and in the reduction of the fact that λ and ρ satisfy equations (V, 22) and (23) respectively. This gives

$$\begin{aligned} - \left(\frac{\partial}{\partial v} \log \rho + \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \right) \frac{\partial x_1}{\partial u} + \left(\frac{\partial}{\partial v} \log \rho + \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' \right) \frac{\partial x_1}{\partial v} \\ + \left(\frac{\partial}{\partial v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' - \frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' \right) \rho \lambda \end{aligned}$$

and similar equations in y_1 and z_1 it follows that a necessary and sufficient condition that the parametric curves form a conjugate system is

$$\frac{\partial}{\partial u} \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' = \frac{\partial}{\partial v} \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}'$$

This condition is satisfied by a system of curves on the surface S if they represent the asymptotic lines on a unique surface S_1 whose coordinates are given by the quadratures (VI, 14)

$$\frac{D'}{H^2} \left(\frac{\partial X}{\partial u} - \frac{\partial Y}{\partial v} \right), \quad \frac{D'}{H^2} \left(-\frac{\partial X}{\partial v} + \frac{\partial Y}{\partial u} \right),$$

and similar expressions for y and z . Combining these equations with (54), we find that

$$\sum \frac{\partial x_i}{\partial u} \frac{\partial x}{\partial u} = 0, \quad \sum \frac{\partial x_i}{\partial u} \frac{\partial x}{\partial v} + \sum \frac{\partial x_i}{\partial v} \frac{\partial x}{\partial u} = 0, \quad \sum \frac{\partial x_i}{\partial v} \frac{\partial x}{\partial v} = 0$$

Hence S and S_i correspond with orthogonality of linear elements, and the normals to the former are parallel to the lines of the congruence. Hence

A necessary and sufficient condition that the developables of a congruence cut the middle surface in a conjugate system is that their representation be that also of the asymptotic lines of a surface, in which case the latter and the middle surface correspond with orthogonality of linear elements

EXAMPLES

When the coördinates of the unit sphere are in the form (III, 35), the parametric curves are asymptotic lines. Find the W -congruences for which the sphere is one of the focal sheets.

2. Let $v_i = f_i(u) + \phi_i(v)$, where f_i and ϕ_i are functions of u and v respectively, and $i = 1, 2, 3$, be three solutions of the first of equations (69), in which case $\lambda = 0$, and let θ_1 in (74) be unity. Show that for the corresponding W -congruence the middle surface is a surface of translation with the generatrices $u = \text{const}$, $v = \text{const}$, that the functions f_i and ϕ_i are proportional to the direction cosines of the binormals to these generatrices, and that the intersections of the osculating planes of these generatrices are the lines of the congruence.

3. Show that isotropic congruences and congruences of Guichard are congruences of Ribaucour.

4. A necessary and sufficient condition that a congruence of Ribaucour be normal is that the spherical representation of its developables be isothermic.

5. The normals to quadrics and to the cyclides of Dupin constitute congruences of Ribaucour.

6. When the middle surface of a congruence is plane the congruence is of the Ribaucour type.

7. Show that the congruence of Ribaucour whose director surface is a skew helicoid, is a normal congruence, and that the normal surfaces are molding surfaces.

8. Show that a necessary and sufficient condition that a congruence of Ribaucour be normal is that the director surface be minimal.

GENERAL EXAMPLES

1. Through each line of a congruence there pass two ruled surfaces of the congruence whose lines of striction lie on the middle surface. Their equation is

$$q^2 - (f + f')^2 + e^2 \int \frac{e du^2 + (f + f') du dv + q dv^2}{c^2 du^2 + 2 \int du dv + dv^2}$$

they are called the *mean ruled surfaces* of the congruence.

that the mean ruled surfaces of a congruence are represented on the orthogonal system of real lines, and that their central planes (§ 105) are the focal planes. Let $u = \text{const.}$, $v = \text{const.}$ be the mean surfaces and develop a theory analogous to that in § 167.

If two focal surfaces of a congruence intersect, the intersection is the edge of regression of the two families of developable surfaces of the congruence.

A congruence consists of the lines joining points on two twisted curves, the focal surfaces of the congruence are determined by the line and the tangent to the focal surface at the point where the curve is met by the line.

Consider that the lines which join the centers of geodesic curvature of the focal surfaces on a surface shall form a normal congruence, it is sufficient that the corresponding radii of geodesic curvature be functions of one another, or that the curves in one family have constant geodesic curvature.

If S be a surface whose lines of curvature in one system are circles, let C be a vertex of the cone circumscribing S along a circle, and L the corresponding generator of the envelope of the planes of the circles, a necessary and sufficient condition that the lines through the points C and the corresponding lines L form a normal congruence is that the distance from C to the points of the corresponding line L be the same for every circle, if this distance be denoted by a , the congruence is given by $R^2 = R^2 + a^2$,

where $\frac{d}{ds}$ indicates differentiation with respect to the arc of the curve of the focal surface.

Let S be a surface referred to its lines of curvature, C_1 and C_2 the centers of normal curvature at a point, G_1 and G_2 the centers of geodesic curvature of the lines of curvature at this point, a necessary and sufficient condition that the lines joining C_1 and C_2 form a normal congruence is that ρ_2 be a function of ρ_1 or that one of these radii be a constant.

Let S be a surface of the first kind defined in Ex. 6. The cone formed by the focal surface at a point of a circle C is tangent to the second sheet of the focal surface (Ex. 1 § 167). Show that the lines through points of C and the corresponding lines L which circumscribe S along C generate a normal congruence and that C lies in the plane of L .

Find a theorem of the first kind system on the sphere for which the linear

$$L^2 = \lambda(L^2 + d^2)$$

is satisfied. Let C be a circle of length λ measured from one extremity of the segment of length λ measured from the other extremity of the segment draw a line parallel to the plane of the circle C . Show that this congruence is isotropic and its direction cosines are of the form

$$\frac{e^2}{e^2 + 1}, \quad \frac{2e}{e^2 + 1}, \quad \frac{e^2}{e^2 + 1}$$

a general integral is

$$f(u) = 2[u - v]f(u)(1 + u)^{-1} + f - \phi,$$

ϕ an arbitrary function of u and v respectively. Find the equations of the focal surface.

- 11 Show that the intersections of the planes

$$(1 - u^2)x + i(1 + u^2)y + 2uz + 4f(u) = 0,$$

$$(1 - v^2)x - i(1 + v^2)y + 2vz + 4\phi(v) = 0$$

constitute an isotropic congruence, for which these are the focal planes, that locus of the mid-points of the lines joining points on the edges of regression of developables enveloped by these planes is the minimal surface which is the middle envelope of the congruence, by finding the coördinates of the point in which the tangent plane to this surface meets the intersection of the above planes

- 12 Show that the middle surface of an isotropic congruence is the most general surface which corresponds to a sphere with orthogonality of linear elements, that the corresponding associate surface in the infinitesimal deformation of a sphere is the minimal surface adjoint to the middle envelope

- 13 Find the surface associate to the middle surface of an isotropic congruence when the surface corresponding to the latter with orthogonality of linear elements is a sphere, and show that it is the polar reciprocal, with respect to the imaginary sphere $x^2 + y^2 + z^2 + 1 = 0$, of the minimal surface adjoint to the middle envelope of the congruence

- 14 The lines of intersection of the osculating planes of the generatrices of a surface of translation constitute a W -congruence of which the given surface is the middle surface, if the generatrices be curves of constant torsion, equal but of opposite sign, the congruence is normal to a W -surface of the type (VIII, 72)

- 15 If the points of a surface S be projected orthogonally upon any plane and if, after the latter has been rotated about any line normal to it through a right angle, lines be drawn through points of A parallel to the corresponding normals to S , these lines form a congruence of Ribaucour

- 16 A necessary and sufficient condition that the tangents to the curves $v = c$ on a surface, whose point equation is (VI, 26), shall form a congruence of Ribaucour

$$\frac{\partial a}{\partial u} \frac{\partial b}{\partial v} + \frac{\partial^2 \log b}{\partial u \partial v} = 0$$

- 17 Show that the tangents to each system of parametric curves on a surface form congruences of Ribaucour when the point equation is

$$\frac{\partial^2 \theta}{\partial u \partial v} + U_1 V_1 \frac{\partial \theta}{\partial u} + U_1 V_1 \frac{\partial \theta}{\partial v} = 0,$$

where U_1 and V_1 are functions of u and v respectively, and the accents indicate differentiation

- 18 Show that if the parametric curves on a surface S form a conjugate system and the tangents to the curves of each family form a congruence of Ribaucour, the same is true of the surfaces S_1 and S_{-1} which together with S constitute the surfaces of the two congruences

- 19 Show that the parameter of distribution p of the ruled surface of a congruence, determined by a value of dv/du , is given by

$$p = \frac{1}{\sqrt{d\sigma^2}} \left| \begin{array}{cc} e du + f dv, & f du + g dv \\ e du + f dv, & f du + g dv \end{array} \right|$$

show that the mean ruled surfaces (of Ex. 1) of a congruence are characterized by the property that for these surfaces the parameter of distribution maximum and minimum values.

S and S_0 are two associate surfaces, and through each point of one a line parallel to the corresponding radius vector of the other, the developables of hence thus formed correspond to the common conjugate system of S and S_0 .

In order that two surfaces S and S_0 corresponding with parallelism of planes be associate surfaces, it is necessary and sufficient that for the surface formed by the joins of corresponding points M and M_0 of these surfaces the developables cut S and S_0 in their common conjugate system, and that points M and M_0 form a harmonic range.

In order that a surface S be isothermic, it is necessary and sufficient that it be a congruence of Ribaucour of which S is the middle surface, such that the developables cut S in its lines of curvature.

CHAPTER XIII

CYCLIC SYSTEMS

174 General equations of cyclic systems The term *congruence* is not restricted to two-parameter systems of straight lines, but applied to two-parameter systems of any kind of curves. Darb has made a study of these general congruences and Ribaucour considered congruences of plane curves. Of particular interest is the case where these curves are circles. Ribaucour has given the name *cyclic systems* to congruences of circles which admit of a one-parameter family of orthogonal surfaces. This chapter is devoted to a study of cyclic systems.

We begin with the general case where the planes of the circles envelop a nondevelopable surface S . We associate with the surface a moving trihedral (§ 68), and for the present assume that the parametric curves on the surface are any whatever.

As the circles lie in the tangent planes to S , the coordinates of a point on one of them with respect to the corresponding trihedral are of the form

$$(1) \quad a + R \cos \theta, \quad b + h \sin \theta, \quad 0,$$

where a, b are the coordinates of the center, h the radius, and θ the angle which the latter to a given point makes with the x -axis.

In § 69 we found the following expressions for the projections of a displacement of a point with respect to the moving axes:

$$(2) \quad \begin{cases} dx + \xi du + \xi_1 dv + (q du + q_1 dv)z - (r du + r_1 dv)y \\ dy + \eta du + \eta_1 dv + (r du + r_1 dv)x - (p du + p_1 dv)z \\ dz + (p du + p_1 dv)y - (q du + q_1 dv)x, \end{cases}$$

* Vol II pp 1-10 also Eisenhart Congruences of Curves *Transactions of the Math Soc*, Vol IV (1903) pp 470-488

† Mémoire sur la théorie générale des surfaces courbes *Journal de Mathématiques* Ser 4, Vol VII (1891), § 117 et seq

the translations ξ, ξ_1, η, η_1 and the rotations p, q, r, p_1, q_1, r_1 the conditions

$$\begin{cases} \frac{\partial p}{\partial v} - \frac{\partial p_1}{\partial u} = qr_1 - r q_1, & \frac{\partial \xi}{\partial v} - \frac{\partial \xi_1}{\partial u} = \eta r_1 - \eta_1 r, \\ \frac{\partial q}{\partial v} - \frac{\partial q_1}{\partial u} = rp_1 - p r_1, & \frac{\partial \eta}{\partial v} - \frac{\partial \eta_1}{\partial u} = \xi_1 r - \xi r_1, \\ \frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u} = pq_1 - q p_1, & p \eta_1 - p_1 \eta = q \xi_1 - q_1 \xi \end{cases}$$

when the values (1) are substituted in (2) the latter are able to

$$\begin{cases} A du + A_1 dv + \cos \theta dR - (d\theta + r du + r_1 dv) R \sin \theta, \\ B du + B_1 dv + \sin \theta dR + (d\theta + r du + r_1 dv) R \cos \theta, \\ (p du + p_1 dv)(h + R \sin \theta) - (q du + q_1 dv)(a + R \cos \theta), \end{cases}$$

so we have put, for the sake of brevity,

$$\begin{cases} I = \frac{ca}{cu} + \xi - rh, & B = \frac{\partial h}{\partial u} + \eta + ra, \\ I_1 = \frac{ca}{cv} + \xi_1 - r_1 h, & B_1 = \frac{\partial h}{\partial v} + \eta_1 + r_1 a \end{cases}$$

conditions that

$$\frac{\partial}{\partial u} \left(\frac{ca}{cu} \right) - \frac{\partial}{\partial v} \left(\frac{ca}{cv} \right), \quad \frac{\partial}{\partial u} \left(\frac{cb}{cu} \right) - \frac{\partial}{\partial v} \left(\frac{cb}{cv} \right)$$

reducible by means of (3) to

$$\begin{cases} \frac{\partial I}{\partial u} - \frac{\partial I_1}{\partial v} = r_1 B - r B_1 - h(p q_1 - p_1 q), \\ \frac{\partial B}{\partial u} - \frac{\partial B_1}{\partial v} = r_1 I + r I_1 + a(p q_1 - p_1 q) \end{cases}$$

the direction cosines of the tangent to the given circle at the point (1) are

$$\sin \theta, \quad \cos \theta, \quad 0$$

so the condition that the locus of the point as u and v vary, be orthogonal to the circle is that the sum of the expressions (4) multiplied respectively by the quantities (7) be zero. This gives

$$R d\theta + (B \cos \theta - I \sin \theta + h r) du + (B_1 \cos \theta - I_1 \sin \theta + h r_1) dv = 0$$

In order that the system of circles be normal to a family of surfaces this equation must admit of a solution involving a parameter. Since it is of the form

$$(9) \quad R d\theta + U du + V dv = 0,$$

the condition that such an integral exist is that the equation

$$(10) \quad R \left(\frac{\partial U}{\partial v} - \frac{\partial V}{\partial u} \right) + U \left(\frac{\partial V}{\partial \theta} - \frac{\partial R}{\partial v} \right) + V \left(\frac{\partial R}{\partial u} - \frac{\partial U}{\partial \theta} \right) = 0$$

be satisfied identically *. For equation (8) this condition reducible to

$$(11) \quad \begin{cases} \sin \theta \left[\frac{\partial R}{\partial v} A - \frac{\partial R}{\partial u} A_1 + Rb(pq_1 - p_1q) \right] \\ - \cos \theta \left[\frac{\partial R}{\partial v} B - \frac{\partial R}{\partial u} B_1 - Ra(pq_1 - p_1q) \right] \\ + (AB_1 - A_1B) + R^2(pq_1 - p_1q) = 0 \end{cases}$$

In order that this equation be satisfied identically, the expressions in the brackets must be zero. If they are not zero, it is possible that the two solutions of this equation will satisfy (8), and determine two surfaces orthogonal to the congruence of circles. Hence we have the theorem of Ribaucour

If the circles of a congruence are normal to more than two surfaces they form a cyclic system

The equations of condition that the system be cyclic consequently

$$(12) \quad \begin{cases} \frac{\partial R}{\partial v} A - \frac{cR}{cu} A_1 + Rb(pq_1 - p_1q) = 0, \\ \frac{\partial R}{\partial v} B - \frac{cR}{cu} B_1 - Ra(pq_1 - p_1q) = 0, \\ (AB_1 - A_1B) + R^2(pq_1 - p_1q) = 0 \end{cases}$$

The total curvature of S is given by (cf § 70)

$$K = \frac{pq_1 - p_1q}{\xi\eta_1 - \xi_1\eta}$$

* Murray *Differential Equations* p 137 New York, 1897 also Forsyth, *Differential Equations*, p 257 London, 1888

is and (5) it is seen that equations (12) involve only s relating to the linear element of S and to the circle we have the theorem of Ribaucour.

envelope of the planes of the circles of a cyclic system be $\bar{S}$ in any manner without disturbing the size or position of S relative to the point of contact, the congruence of circles s to form a cyclic system

Moreover, if we put

$$t = \tan \frac{\theta}{2},$$

(8) assumes the Riccati form,

$$dt + (a_1 t^2 + a_2 t + a_3) du + (b_1 t^2 + b_2 t + b_3) dv = 0,$$

the a 's and b 's are functions of u and v . Recalling a fundamental property of such equations (§ 14), we have

four orthogonal surfaces of a cyclic system meet the circles in curves whose cross ratio is constant

by hypothesis S is nondevelopable, equations (12) may be reduced by

$$\begin{cases} h \frac{cR}{cu} = 1a + Bh, \\ h \frac{cH}{cu} = 1_1 a + B_1 b, \\ 1h_1 = 1_1 b + h^2 (\xi \eta_1 - \xi_1 \eta) h = 0 \end{cases}$$

the first two of these equations are reducible to

$$\begin{cases} \frac{c}{cu} (h^2 - a - b^2) = 2(a\xi + b\eta), \\ \frac{c}{cu} (h - a - b) = 2(a\xi_1 + b\eta_1) \end{cases}$$

condition of integrability of these equations is

$$\frac{ca}{cu} \xi + \frac{cb}{cu} \eta - \frac{ca}{cu} \xi_1 - \frac{cb}{cu} \eta_1 = r(a\eta_1 - b\xi_1) - r_1(a\eta - b\xi)$$

in considering this equation, we introduce a function ϕ by the equation

$$2\phi = R^2 - a^2 - b^2,$$

CYCLIC SYSTEMS

the condition which ϕ must satisfy. We take for expressions obtained by solving (14), that is

$$a = \frac{\eta_1 \frac{\partial \phi}{\partial u} - \eta \frac{\partial \phi}{\partial v}}{\xi \eta_1 - \xi_1 \eta}, \quad b = \frac{-\xi_1 \frac{\partial \phi}{\partial u} + \xi \frac{\partial \phi}{\partial v}}{\xi \eta_1 - \xi_1 \eta}$$

the equation (15) vanishes identically, and the only other condition to be satisfied is the last of (13), thus, by the substitution of these values of a , b , R , becomes a partial differential equation in ϕ of the form

$$(18) \quad \frac{\partial^2 \phi}{\partial u^2} \frac{\partial^2 \phi}{\partial v^2} - \left(\frac{\partial^2 \phi}{\partial u \partial v} \right)^2 + J \frac{\partial^2 \phi}{\partial u^2} + L \frac{\partial^2 \phi}{\partial u \partial v} + M \frac{\partial^2 \phi}{\partial v^2} + N = 0,$$

where J , L , M , N denote functions of ϕ , ξ , η , and their derivatives of the first order. Conversely, each solution of this equation gives a cyclic system whose circles lie in the tangent planes to S .

EXAMPLES

1. Let S be a surface of revolution defined by (III, 90), and let T be the trihedral whose z axis is tangent to the curve $v = \text{const}$. Determine the condition which the function $\psi(u)$ must satisfy in order that the quantities a , b in (1) may have the values

$$a = \frac{\psi(u)}{\sqrt{1 + \phi'^2(u)}}, \quad b = \frac{1}{u}$$

and determine also the expression for R .

2. A necessary and sufficient condition that all the circles of a cyclic system whose planes envelop a nondevelopable surface shall have the same radius, is that the planes of the circles touch their envelope S at the centers of the circles, and that S be pseudospherical.

3. Let S be a surface referred to an orthogonal system of lines and let T be the trihedral whose z axis is tangent to the curve $v = \text{const}$. With reference to the trihedral the equations of a curve in the tangent plane are of the form

$$x = \rho \cos \theta, \quad y = \rho \sin \theta, \quad z = 0$$

where in general ρ is a function of θ , u and v . Show that the condition that there be a surface orthogonal to these curves is that there exist a relation between θ , u and v which satisfies the equation

$$\left[\rho^2 + \left(\frac{\partial \rho}{\partial \theta} \right)^2 \right] d\theta + \left(\frac{\partial \rho}{\partial \theta} \frac{\partial \rho}{\partial u} + \xi \cos \theta \frac{\partial \rho}{\partial \theta} - \rho \xi \sin \theta + \rho^2 r \right) du + \left(\frac{\partial \rho}{\partial \theta} \frac{\partial \rho}{\partial v} + \eta_1 \sin \theta \frac{\partial \rho}{\partial \theta} + \rho \eta_1 \cos \theta + \rho^2 r_1 \right) dv = 0$$

When this condition is satisfied by a function θ which involves an arbitrary constant, there is an infinity of normal surfaces. In this case the curves are said to form a *normal congruence*.

4. When the surface enveloped by the planes of the curves of a normal congruence of plane curves is deformed in such a way that the curves remain invariably fixed to the surface, the congruence continues to be normal.

cyclic congruences The axes of the circles of a cyclic system constitute a rectilinear congruence which Bianchi* has called *congruence*. In order to derive the properties of this congruence and further results concerning cyclic systems, we assume that the parametric curves on S correspond to the developables of the congruence.

The coordinates of the focal points of a line of the congruence and the corresponding trihedral are of the form a, b, ρ . On the hypothesis that the former are the coordinates of the focal point for the developable $v = \text{const}$ of the line, we have, from (2),

$$\frac{\partial a}{\partial u} + \xi + q\rho_1 - r b = 0, \quad \frac{\partial b}{\partial u} + \eta - p\rho_1 + r a = 0$$

and in like manner with the other point, we obtain a pair of similar equations. All of these equations may be written in the following form

$$q\rho_1 = 0, \quad B - p\rho_1 = 0, \quad l_1 + q_1\rho_2 = 0, \quad B_1 - p_1\rho_2 = 0,$$

consequence of (5). When these values are substituted in the equations (13), it is found that

$$h^2 = -\rho_1\rho_2$$

lines joining a point on the circle to the focal points are equal. If we put

$$2\rho = \rho_1 + \rho_2, \quad 2\delta = \rho_1 - \rho_2,$$

denoting by 2ρ the distance between the focal points, and by δ the distance between the center of the circle and the mid-point of the congruence, we find that

$$h^2 + \delta^2 = \rho^2$$

and this equation by the two

$$c = \rho \cos \sigma, \quad h = \rho \sin \sigma,$$

regarding σ as a function of ρ . Now we have

$$\rho_1 = \rho(\cos \sigma + 1), \quad \rho_2 = \rho(\cos \sigma - 1),$$

and the equations (14) may be written

$$\begin{cases} 1 = q\rho(\cos \sigma + 1), & B = p\rho(\cos \sigma + 1), \\ 1 = -q_1\rho(\cos \sigma - 1), & B_1 = p_1\rho(\cos \sigma - 1) \end{cases}$$

By means of (5) equation (15) can be put in the form

$$A_1\xi - A\xi_1 + B_1\eta - B\eta_1 = 0$$

When the values (22) are substituted in this equation, it becomes

$$\cos \sigma (p\eta_1 - p_1\eta - q\xi_1 + q_1\xi) + (p_1\eta - q_1\xi + p\eta_1 - q\xi_1) = 0$$

Since by (3) the expression in the first parenthesis is zero, the same is true of the second, and so we have

$$p\eta_1 - q\xi_1 = 0, \quad p_1\eta - q_1\xi = 0$$

But these are the conditions (V, 67) that the parametric curves on S form a conjugate system. Hence we have the theorem of Ribaucour

On the envelope of the planes of the circles of a cyclic system the curves corresponding to the developables of the associated cyclic congruence form a conjugate system

176 Spherical representation of cyclic congruences When the expressions (22) are substituted in (6), we obtain

$$\begin{aligned} q_1 \frac{\partial}{\partial u} [\rho (\cos \sigma - 1)] - q \frac{\partial}{\partial v} [\rho (\cos \sigma + 1)] \\ = \rho \left(\frac{\partial q}{\partial v} + \frac{\partial q_1}{\partial u} + r_1 p + r p_1 \right) - b (p q_1 - p_1 q), \\ p_1 \frac{\partial}{\partial u} [\rho (\cos \sigma - 1)] - p \frac{\partial}{\partial v} [\rho (\cos \sigma + 1)] \\ = \rho \left(\frac{\partial p}{\partial v} + \frac{\partial p_1}{\partial u} - r_1 q - r q_1 \right) - a (p q_1 - p_1 q) \end{aligned}$$

Since $p q_1 - p_1 q \neq 0$ unless S is developable, the preceding equation may be replaced by

$$(23) \quad \begin{cases} \frac{\partial}{\partial u} [\rho (\cos \sigma - 1)] = 2 \rho \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' + (a q - b p), \\ \frac{\partial}{\partial v} [\rho (\cos \sigma + 1)] = -2 \rho \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' + (a q_1 - b p_1), \end{cases}$$

where the Christoffel symbols are formed with respect to

$$(24) \quad (p du + p_1 dv)^2 + (q du + q_1 dv)^2,$$

the linear element of the spherical representation of S

in like manner we substitute in the first two of equations

$$R^2 = \rho^2 \sin^2 \sigma = \rho^2 (1 - \cos^2 \sigma),$$

$$1 - \cos \sigma \left\{ \frac{\partial \rho}{\partial u} - \frac{\rho \cos \sigma}{1 + \cos \sigma} \frac{\partial}{\partial u} \cos \sigma \right\} = p b - q a,$$

$$1 + \cos \sigma \left\{ \frac{\partial \rho}{\partial v} - \frac{\rho \cos \sigma}{1 - \cos \sigma} \frac{\partial}{\partial v} \cos \sigma \right\} = -p_1 b + q_1 a$$

in equations and (28) we find

$$\begin{cases} \frac{\partial}{\partial u} \cos \sigma = 2(\cos \sigma + 1) \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}', \\ \frac{\partial}{\partial v} \cos \sigma = 2(\cos \sigma - 1) \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}', \end{cases}$$

$$\begin{cases} (\cos \sigma - 1) \frac{\partial \rho}{\partial u} = -2 \rho \cos \sigma \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' + q a - p b, \\ (\cos \sigma + 1) \frac{\partial \rho}{\partial v} = -2 \rho \cos \sigma \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' + q_1 a - p_1 b \end{cases}$$

dition of . . . of equations (25) is reducible to

$$\left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' - \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \cos \sigma = \frac{\partial}{\partial u} \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' + \frac{\partial}{\partial v} \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' - 4 \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}'$$

ression for $\cos \sigma$ obtained from this equation be substituted, we find two conditions upon the curves on the sphere at they may represent the developables of a cyclic congruence. A particular case is that in which (27) is identically satisfied the two conditions are

$$\frac{\partial}{\partial u} \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' - \frac{\partial}{\partial v} \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' - 2 \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}'$$

we now purpose to show that if any system of curves on the sphere satisfies either set of conditions, all the congruences developables on the sphere are cyclic congruences. We assume that the sphere is referred to such a system and that the solution ρ of

$$+ \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' \frac{\partial \rho}{\partial u} + \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' \frac{\partial \rho}{\partial v} + \left(\frac{\partial}{\partial u} \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' + \frac{\partial}{\partial v} \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' + \mathcal{D} \right) \rho = 0$$

By the method of § 167, or that hereafter explained, we find the middle surface of the congruence. Then we take the point on each line at the distance $-\rho \cos \sigma$ from the mid-point as the center of the circle of radius $\rho \sin \sigma$ and for which the line is the axis. These circles form a cyclic system, as we shall show.

In the first place we determine the middle surface with reference to a trihedral of fixed vertex, whose z -axis coincides with the radius of the sphere parallel to the line of the congruence and whose x - and y -axes are any whatever. If x_0, y_0, z_0 denote the coördinates of the mid point of a line with reference to the corresponding trihedral, the coordinates of the focal points are

$$x_0, \quad y_0, \quad z_0 + \rho, \quad x_0, \quad y_0, \quad z_0 - \rho$$

From (2) it is seen that if these points correspond to the developables $v = \text{const.}$ and $u = \text{const.}$ respectively, we must have

$$\begin{aligned} \frac{\partial x_0}{\partial u} + q(z_0 + \rho) - r y_0 &= 0, & \frac{\partial y_0}{\partial u} + r x_0 - p(z_0 + \rho) &= 0, \\ \frac{\partial x_0}{\partial v} + q_1(z_0 - \rho) - r_1 y_0 &= 0, & \frac{\partial y_0}{\partial v} + r_1 x_0 - p_1(z_0 - \rho) &= 0 \end{aligned}$$

Since $p q_1 - p_1 q \neq 0$, the conditions of integrability of these equations can be put in the form

$$(30) \quad \begin{cases} \frac{\partial z_0}{\partial u} - \frac{\partial \rho}{\partial u} - 2\rho \left\{ \begin{smallmatrix} 12 \\ 2 \end{smallmatrix} \right\}' + (p_1 y_0 - q_1 x_0) = 0, \\ \frac{\partial z_0}{\partial v} + \frac{\partial \rho}{\partial v} + 2\rho \left\{ \begin{smallmatrix} 12 \\ 1 \end{smallmatrix} \right\}' + (p_1 y_0 - q_1 x_0) = 0 \end{cases}$$

It is readily found that the condition of integrability of these equations is reducible to (29)

It will be to our advantage to have also the coordinates of the point of contact of the plane of the circle with its envelope S . If $x, y, z_0 - \rho \cos \sigma$ denote these coordinates with reference to the above trihedral, it follows from (2) that

$$\begin{aligned} \frac{\partial}{\partial u} (z_0 - \rho \cos \sigma) + p y - q_1 x &= 0, \\ \frac{\partial}{\partial v} (z_0 - \rho \cos \sigma) + p_1 y - q_1 x &= 0 \end{aligned}$$

ations be subtracted from the respective ones of (30), are reducible, by means of (25), to

$$-1) \frac{\partial \rho}{\partial u} + 2 \rho \cos \sigma \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}' + p(y_0 - \bar{y}) - q(x_0 - \bar{x}) = 0,$$

$$+1) \frac{\partial \rho}{\partial v} + 2 \rho \cos \sigma \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}' + p_1(y_0 - \bar{y}) - q_1(x_0 - \bar{x}) = 0,$$

the same as (26). For, the quantities $x_0 - \bar{x}$, $y_0 - \bar{y}$ are ones of the center of the circle with reference to the tri-parallel to the preceding one and with the corresponding for vertex

we have a solution σ of (25) and ρ of (29), the corre-values of a and b given by (26) satisfy (22), since the he conditions that the parametric curves on the sphere he developables of the congruence. However, we have when the values (22) are substituted in (12), we obtain educible to (25) and (26). Hence the circles constructed l above form a cyclic system

uations (25) admit only one solution (27) unless the con-is satisfied, we have the theorem

h cyclic congruence there is associated a unique cyclic sys-it is at the same time a congruence of Ribaucour, in which s an infinity of associated cyclic systems

g the results of § 141, we have the theorem of Bianchi *
e total curvature of a surface referred to its asymptotic he form

$$K = \frac{1}{[\phi(u) + \psi(v)]^2},$$

urface generates of a congruence of Ribaucour which is n infinity of ways and these are the only cyclic congru-in infinity of associated cyclic systems

use the general solution of equations (25) is

$$\cos \sigma = \frac{\psi - \phi + a}{\phi + \psi},$$

an arbitrary constant

177. Surfaces orthogonal to a cyclic system In this section we consider the surfaces S_1 orthogonal to the circles of a cyclic system. Since the direction-cosines of the normals to the surfaces with reference to the moving trihedral in § 174 are $-\sin \theta, \cos \theta, 0$, the spherical representation of these surfaces is given by the point whose coordinates are these with respect to a trihedral of fixed vertex parallel to the above trihedral. From (2) we find that the expressions for the projections of a displacement of this point are

$$\begin{aligned} & -\cos \theta (d\theta + r du + r_1 dv), \\ & -\sin \theta (d\theta + r du + r_1 dv), \\ & (p du + p_1 dv) \cos \theta + (q du + q_1 dv) \sin \theta \end{aligned}$$

Moreover, by means of (8), (21), (22), we obtain the identity

$$(32) \quad \sin \sigma (d\theta + r du + r_1 dv) = -(1 + \cos \sigma) (p \cos \theta + q \sin \theta) du \\ + (1 - \cos \sigma) (p_1 \cos \theta + q_1 \sin \theta) dv$$

Hence the linear element of the spherical representation of S_1 is

$$(33) \quad d\sigma_1^2 = \frac{2}{1 - \cos \sigma} (p \cos \theta + q \sin \theta)^2 du^2 \\ + \frac{2}{1 + \cos \sigma} (p_1 \cos \theta + q_1 \sin \theta)^2 dv^2$$

Since the parametric curves on the sphere form an orthogonal system, the parametric curves on the surface are the lines of curvature, if they form an orthogonal system. In order to show that this condition is satisfied, we first reduce the expressions (4) for the projections of a displacement of a point on S_1 , by means of (21), (22), (25), (26), and (32), to

$$(34) \quad \begin{cases} \cos \theta \sin \sigma \left(\frac{C' du}{1 - \cos \sigma} - \frac{D' dv}{1 + \cos \sigma} \right), \\ \sin \theta \sin \sigma \left(\frac{C' du}{1 - \cos \sigma} - \frac{D' dv}{1 + \cos \sigma} \right), \\ C' du + D' dv, \end{cases}$$

where we have put

$$(35) \quad \begin{cases} C = p(b + R \sin \theta) - q(a + R \cos \theta), \\ D = p_1(b + R \sin \theta) - q_1(a + R \cos \theta) \end{cases}$$

a linear element of S_1 is

$$ds_1^2 = 2 \frac{C^2 dv^2}{1 - \cos \sigma} + 2 \frac{D^2 dv^2}{1 + \cos \sigma},$$

wh it is seen that the parametric curves on S_1 form an orthogonal system, and consequently are the lines of curvature. Furthermore, it is seen from (84) that the tangents to the curves $\sigma = \text{const.}$ make with the plane of the circle the respective angles

$$\tan^{-1} \left(\frac{1 - \cos \sigma}{\sin \sigma} \right), \quad \tan^{-1} \left(-\frac{1 + \cos \sigma}{\sin \sigma} \right)$$

It follows from (21) that the lines joining a point on the circle of a circle to the focal points of its axis make the same angle with the radius to the point. Hence we have

The tangents to the lines of curvature on a surface orthogonal to a cyclic system are tangents to the developables of the congruence of axes of the circles, and the tangents to the two lines of curvature through a point of the surface meet the corresponding axis in its focal points.

Normal cyclic congruences Since the developables of a congruence correspond to a conjugate system on the envelope of the planes of the circles, this system consists of the lines of curvature when the congruence is normal, and only in this case (cf § 83). If, under these conditions, we take two of the lines of the trihedral tangent to the lines of curvature, we have

$$\xi_1 = \eta = 0, \quad p = q = 0,$$

and the relations (25) become

$$u \log \cos \frac{\sigma}{2} = v \log p, \quad \frac{\partial}{\partial u} \log \sin \frac{\sigma}{2} = \frac{\partial}{\partial v} \log q$$

For a suitable choice of parameters we have

$$p = \cos \frac{\sigma}{2}, \quad q = -\sin \frac{\sigma}{2},$$

we put $\omega = \sigma/2$ the linear element of the sphere is

$$d\sigma^2 = \sin^2 \omega du^2 + \cos^2 \omega dv^2$$

Combining this result with (§ 119), we have the theorem

Normals to a surface Σ with the same spherical representation as of curvature as a pseudospherical surface constitute the envelope of normal cyclic congruences.

Since the surface Σ and the envelope S of the planes of the circles have the same representation of their lines of curvature, the tangents to the latter at corresponding points on the two surfaces are parallel. Hence with reference to a trihedral for Σ parallel to the trihedral for S the coordinates of a point on the circle are $R \cos \theta$, $R \sin \theta$, μ , where μ remains to be determined and θ is given by (32), which can be put in the form

$$(40) \quad \frac{\partial \theta}{\partial u} + \frac{\partial \omega}{\partial v} = \cos \omega \sin \theta, \quad \frac{\partial \theta}{\partial v} + \frac{\partial \omega}{\partial u} = -\sin \omega \cos \theta$$

If we express, by means of (2), the condition that all displacements of this point be orthogonal to the line whose direction cosines are $-\sin \theta$, $\cos \theta$, 0, the resulting equation is reducible by means of (40), to

$$\begin{aligned} & \sin \theta (R \cos \omega - \mu \sin \omega - \xi) du \\ & - \cos \theta (R \sin \omega + \mu \cos \omega - \eta_1) dv = 0 \end{aligned}$$

Hence the quantities in parentheses are zero, from which we obtain

$$(41) \quad R = \xi \cos \omega + \eta_1 \sin \omega, \quad \mu = -\xi \sin \omega + \eta_1 \cos \omega$$

When, in particular, Σ is a pseudospherical surface of curvature $-1/a^2$, we have (VIII, 22)

$$\xi = a \cos \omega, \quad \eta_1 = a \sin \omega,$$

so that $R = a$ and $\mu = 0$. Hence the circles are of constant radius and the envelope of their planes is the locus of their center (cf Ex 2, § 174). Conversely, when the latter condition is satisfied, it follows from (13) that R is constant. Moreover, in the case ρ_1 and ρ , as defined in § 175, are the principal radii of the surface, which by (20) is pseudospherical. When these values are substituted in (36) and (33), it is found that the linear element of each orthogonal surface is

$$ds_1^2 = a^2 (\cos^2 \theta du^2 + \sin^2 \theta dv^2),$$

and of its spherical representation

$$(42) \quad d\sigma_1^2 = \sin^2 \theta du^2 + \cos^2 \theta dv^2$$

Hence these orthogonal surfaces are the transforms of Σ by means of the Bianchi transformation (§ 119)

ression (42) is the linear element of the spherical representation of the surfaces orthogonal to the circles associated with Σ , whether it be pseudospherical or not, whose spherical representation is given by (39). Since these orthogonal surfaces are of the same type as Σ . We have thus for all surfaces with the same representation of their lines of curvature as pseudospherical surfaces, the representation into similar surfaces of which the Bianchi transformation is a particular case, we call it a generalized Bianchi transformation.*

Special systems for which the envelope of the planes of the curve We consider now the particular cases which have resulted from the preceding discussion, and begin with that in which the envelope S of the planes of the circles is a curve C . We consider the moving trihedral such that its xy -plane, as before, is tangent to the circle, and take the x -axis tangent to C . If s denotes the arc length along C , we have

$$ds = \xi du + \xi_1 dv, \quad \eta = \eta_1 = 0,$$

$$r\xi_1 - r_1\xi = 0, \quad q\xi_1 - q_1\xi = 0$$

(15), and (16) it follows that a and ϕ are functions of s , and the equations may be replaced by

$$h^2 = a^2 + b^2 + 2\phi(s)$$

parametric curves on the sphere represent the developables generated by the conditions (19) must hold. But from (5), (13) we obtain

$$1\xi_1 - 1_1\xi = 0$$

and from (19) be substituted in this equation, we have,

$$\rho_1 - \rho = 0$$

if the focal surfaces coincide. If we put

$$\rho = \rho_1 = \rho_2$$

and substitute in the last of (12), we obtain

$$(\rho^2 + h^2)(pq_1 - p_1q) = 0$$

The vanishing of $pq_1 - p_1q$ is the condition that there be a single infinity of planes, which case we exclude for the present. Hence $\rho = \pm iR$, that is, the developables of the cyclic congruence are imaginary.

Instead of retaining as parametric curves those representing the developables, we make the following choice. We take the arc of C for the parameter u , consequently $\xi = 1$, $\xi_1 = 0$. Since $\eta = \eta_1 = 0$ also, we have, from (3),

$$q_1 = r_1 = 0, \quad \frac{\partial p}{\partial v} - \frac{\partial p_1}{\partial u} = 0,$$

hence we may choose the parameter v so that $p = 0$, $p_1 = 1$. From (3) it follows, furthermore, that

$$\frac{\partial q}{\partial v} = r, \quad \frac{\partial r}{\partial v} = -q,$$

of which the general integral is

$$q = U_1 \cos v + U_2 \sin v, \quad r = -U_1 \sin v + U_2 \cos v,$$

where U_1 and U_2 are arbitrary functions of u . From (5) we have

$$A = \phi''(u) + 1 - rb, \quad A_1 = 0, \quad B_1 = \frac{\partial b}{\partial v},$$

so that the third of equations (12) is reducible by (44) to

$$(45) \quad \frac{(\phi'' + 1) \frac{\partial b}{\partial v}}{(b^2 + \phi'^2 + 2\phi)^{\frac{3}{2}}} = \frac{\partial}{\partial v} \frac{U_1 \sin v - U_2 \cos v}{(b^2 + \phi'^2 + 2\phi)^{\frac{1}{2}}}$$

Hence if we take for a any function of u denoted by $\phi'(u)$, equation (45) gives b , and R follows directly from (44).

180 Cyclic systems for which the planes of the circles pass through a point. If the planes of the circles of a cyclic system pass through a point O , we take it for the origin and for the vertex of a moving trihedral whose z -axis is parallel to the axis of the circle under consideration. In this case equations (14) may be replaced by

$$(46) \quad R^2 = a^2 + b^2 - c,$$

where c denotes a constant. But this is the condition that all the circles are orthogonal to a sphere with center at O , or cut it in

ly opposite points, or pass through O , according as c is negative, or zero. Hence we have the theorem

Axes of the circles of a cyclic system pass through a point, are orthogonal to a sphere with its center at the point, or here in opposite points, or pass through the center

metrical considerations we see that the converse of this is true

in (46) is zero all the circles pass through O . Then we have

$$a = -\rho \sin \sigma \cos \theta, \quad b = -\rho \sin \sigma \sin \theta,$$

hence (26) becomes

$$\sigma - 1) \frac{\partial \log \rho}{\partial u} = -2 \cos \sigma \left\{ \frac{12}{2} \right\}' + \sin \sigma (p \sin \theta - q \cos \theta),$$

$$\sigma + 1) \frac{\partial \log \rho}{\partial v} = -2 \cos \sigma \left\{ \frac{12}{1} \right\}' + \sin \sigma (p_1 \sin \theta - q_1 \cos \theta)$$

analogous relations are obtained likewise when we substitute the values in equations (22) and reduce by means of (25) and (32)

(22) the function ρ given by (26) is a solution of (29), or ρ given by (48) is a solution. But the solution θ of (29) is a parameter. Hence we have the theorem of Bianchi*

All the cyclic congruences with the same spherical representation have their developables tangent to a sphere, and the axes of the associated cyclic system pass through a point

Let the line through O and the center of the circle form the axis of the trihedral, equation (11) must admit of the solution $\theta = 0$, and consequently must be of the form

$$\sin \theta L + (\cos \theta + 1)M = 0$$

that this equation admit of a solution other than π , both L and M must be zero and the system cyclic. We combine this with the preceding theorem to obtain the following

A parameter family of circles through a point and orthogonal to a sphere constitute a cyclic system, and the most general spherical representation of the developables of a cyclic congruence is afforded by the axes of such a system of circles †

We consider finally the case where the planes of the circles depend upon a single parameter. If we take for moving axes the tangent, principal normal, and binormal of the edge of regression of these planes and its arc for the parameter u , we have

$$\xi = 1, \quad \xi_1 = \eta = \eta_1 = 0, \quad p_1 = q_1 = r_1 = 0,$$

and comparing (V, 50) with (2), we see that

$$p = -\frac{1}{\tau}, \quad q = 0, \quad r = \frac{1}{\rho},$$

where ρ and τ are the radii of first and second curvature of the edge of regression. Now

$$A = \frac{\partial a}{\partial u} + 1 - \frac{b}{\rho}, \quad A_1 = \frac{\partial a}{\partial v}, \quad B = \frac{\partial b}{\partial u} + \frac{a}{\rho}, \quad B_1 = \frac{\partial b}{\partial v}$$

The equations (12) reduce to two. One of the functions a , b may be chosen arbitrarily, then the other and R can be obtained by the solution of partial differential equations of the first order

EXAMPLES

1 Show that a congruence of Ribaucour whose surface generator is the right helicoid is cyclic, and determine the cyclic systems.

2 A congruence of Guichard is a cyclic congruence, and the envelope of the planes of the circles of each associated cyclic system is a surface of Voss.

3 The surface generator of a cyclic congruence of Ribaucour is an associate surface of the planes of the circles of each associated cyclic system.

4 If S is a surface whose lines of curvature have the same spherical representation as a pseudospherical surface, and S_1 is a transform of S resulting from a generalized Bianchi transformation (§ 178), the tangents to the lines of curvature of S_1 pass through the centers of principal curvature of S .

5 When the focal segment of each line of a cyclic congruence is divided in constant ratio by the center of the circle, the envelope of the planes of the circles is a surface of Voss.

6 The circles of the cyclic system whose axes are normal to the surface Σ , defined in Ex. 11, p. 370, pass through a point, and the surfaces orthogonal to the circles are surfaces of Bianchi of the parabolic type.

7 If the spheres with the focal segments of the lines of a congruence for diameters pass through a point, the congruence is cyclic, and the circles pass through the point.

8 Show that the converse of Ex. 7 is true.

GENERAL EXAMPLES

mine the normal congruences of Ribaucour which are cyclic.

envelope of the planes of the circles of a cyclic system is a surface of conjugate geodesic system corresponds to the developables of the associated congruence, any family of planes cutting the focal segments in coincident perpendicular to them envelop a surface of Voss.

necessary and sufficient condition that a congruence be cyclic is that the focal lines have the same spherical representation as the conjugate lines of a surface. If the focal lines remain conjugate in a deformation of the surface. If the developables of the congruence are real, the focal lines of the surface are imaginary.

planes of the cyclic systems associated with a cyclic congruence of focal lines such that their respective envelopes in such a way that the points of contact of the focal planes corresponding to the same line of the congruence lie on a

spheres described on the focal segments of a congruence as diameters. If the focal lines are orthogonal or in great circles, the congruence is cyclic and it is the fixed sphere orthogonally or in diametrically opposite points.

one draws the circles which are normal to a surface S and which cut the focal lines S_0 in diametrically opposite points or orthogonally, the spheres described on the focal segments of the congruence of axes as diameters cut S_0 in diametrically opposite points or orthogonally.

mine the cyclic systems of equal circles whose planes envelop a developable surface.

Let S be the surface defined in Ex. 14, p. 371, and let S_0 be the sphere with center O and radius r . Draw the circles which are normal to S and which cut S_0 orthogonally or in diametrically opposite points. Show that the envelope of the focal lines of these circles is a normal congruence, and that the focal lines of the normal surfaces are of the form

$$x = \left[\frac{1}{2a} \left\{ a^2 e^{-\frac{\xi}{a}} - (\eta^2 + \kappa) e^{\frac{\xi}{a}} \right\} \cos \theta + \eta \sin \theta \right] X_1 \\ + \left[\frac{1}{2a} \left\{ a^2 e^{-\frac{\xi}{a}} - (\eta^2 + \kappa) e^{\frac{\xi}{a}} \right\} \sin \theta - \eta \cos \theta \right] X_2 + t X_3,$$

where $t = -r \cos \theta + r^2$, according as the circles cut S_0 orthogonally or in diametrically opposite points, and where t is given by

$$dt = \left[\frac{1}{2a} \left\{ a^2 e^{-\frac{\xi}{a}} - (\eta^2 + \kappa) e^{\frac{\xi}{a}} \right\} \cos \theta + \eta \sin \theta \right] \sin \omega du \\ - \left[\frac{1}{2a} \left\{ a^2 e^{-\frac{\xi}{a}} - (\eta^2 + \kappa) e^{\frac{\xi}{a}} \right\} \sin \theta - \eta \cos \theta \right] \cos \omega dv$$

that the surfaces of Ex. 8 are surfaces of Bianchi which have the focal lines as the representation of their lines of curvature as the pseudospherical surfaces defined in Ex. 14, p. 371.

that the surfaces orthogonal to the cyclic system of Ex. 8 are surfaces of focal lines of the parabolic type.

11 Let S be a surface referred to an orthogonal system, and let T be the trihedral whose x -axis is tangent to the curve $v = \text{const}$. The equations

$$x = \rho(1 + \cos \theta), \quad y = 0, \quad z = \rho \sin \theta$$

define a circle normal to S . Show that the necessary and sufficient conditions that the circles so defined form a cyclic system are

$$\xi \frac{\partial \rho}{\partial v} + \rho \eta_1 r = 0, \quad \rho(p r_1 - p_1 r) - q_1 \left(\xi + \frac{\partial \rho}{\partial u} \right) + q \frac{\partial \rho}{\partial v} = 0$$

12 A necessary and sufficient condition that a cyclic system remain cyclic when an orthogonal surface S is deformed is that S be applicable to a surface of revolution and that

$$\rho = \frac{1}{\phi} \left(c - \int \phi \, du \right),$$

where c is a constant and the linear element of S is $ds^2 = du^2 + \phi^2(u) dv^2$ (cf Ex 11)

13 Determine under what conditions the lines of intersection of the planes of the circles of a cyclic system and the tangent planes to an orthogonal surface form a normal congruence

14 Let Σ_1 and Σ_2 be two surfaces orthogonal to a cyclic system, and let M_1 and M_2 be the points of intersection of one of the circles with Σ_1 and Σ_2 . Show that the normals to Σ_1 and Σ_2 at the points M_1 and M_2 meet in a point M equidistant from these points, and show that Σ_1 and Σ_2 constitute the sheets of the envelope of a two-parameter family of spheres such that the lines of curvature on Σ_1 and Σ_2 correspond

15 Let S be the surface of centers of a two-parameter family of spheres of variable radius R , and let Σ_1 and Σ_2 denote the two sheets of the envelope of these spheres. Show that the points of contact M_1 and M_2 of a sphere with these sheets are symmetric with respect to the tangent plane to S at the corresponding point M . Let S be referred to a moving trihedral whose plane $y = 0$ is the plane $M_1 M M_2$, and let the parametric curves be tangent to the x - and y axes respectively. Show that if σ denotes the angle which the radius MM_1 makes with the x -axis of the trihedral, the lines of curvature on Σ_1 are given by

$$\xi \sin \sigma (\sin \sigma p - r \cos \sigma) du^2 + \eta_1 \left(q_1 - \frac{\partial \sigma}{\partial v} \right) dv^2 + \left[\eta_1 \left(q - \frac{\partial \sigma}{\partial u} \right) - \xi \sin \sigma (\cos \sigma r_1 + p \sin \sigma) \right] du dv = 0$$

16 Find the condition that the lines of curvature on Σ_1 and Σ_2 of Ex 15 correspond, and show that in this case these curves correspond to a conjugate system on S

17 Show that the circles orthogonal to two surfaces form a cyclic system, provided that the lines of curvature on the two surfaces correspond

18 Let S be a pseudospherical surface with the linear element (VIII, 22), the lines of curvature being parametric, and let A be a surface with the same spherical representation of its lines of curvature as S , furthermore, let A_1 denote the envelope of the plane which makes the constant angle σ with the tangent plane at a point M of A and meets this plane in a line L , which forms with the tangent to the curve $v = \text{const}$ at M an angle θ defined by equations (VIII, 35). If M_1

point of contact of this plane, we drop from M_1 a perpendicular on L , latter in N . Show that if λ and μ denote the lengths MN and NM_1 , an by

$$\sqrt{E} \cos \omega + \sqrt{G} \sin \omega) \sin \sigma, \quad \mu = (-\sqrt{E} \sin \omega + \sqrt{G} \cos \omega) \sin \sigma,$$

1 G are the first fundamental coefficients of A

that when the surface A in Ex 18 is the pseudospherical surface S , the Backlund transform S_1 of S by means of the functions (θ, σ) , and 1 is other than S the lines of curvature on the four surfaces S, A, S_1 , and, and the last two have the same spherical representation

that as θ is given all values satisfying equations (VIII, 35) for a given of the point M_1 , defined in Ex 18, is a circle whose axis is normal to A at M

that when A in Ex 18 is a surface of Bianchi of the parabolic type (370) the surfaces A_1 are of the same kind, whatever be σ

CHAPTER XIV

TRIPLY ORTHOGONAL SYSTEMS OF SURFACES

181 Triple system of surfaces associated with a cyclic system
 Let S_1 be one of the surfaces orthogonal to a cyclic system, and let its lines of curvature be parametric. The locus Σ_1 of the circles which meet S_1 in the line of curvature $v = \text{const}$ through a point M is a surface which cuts S_1 orthogonally. Hence, by Joachimsthal's theorem (§ 59), the line of intersection is a line of curvature for Σ_1 . In like manner, the locus Σ_2 of the circles which meet S_1 in the line of curvature $u = \text{const}$ through M cuts S_1 orthogonally, and the curve of intersection is a line of curvature on Σ_2 also. Since the developables of the associated cyclic congruence correspond to the lines of curvature on all of the orthogonal surfaces, each of the latter is met by Σ_1 and Σ_2 in a line of curvature of both surfaces. At each point of the circle through M the tangent to the circle is perpendicular to the line of curvature $v = \text{const}$ on Σ_1 through the point and to $u = \text{const}$ on Σ_2 . Hence the circle is a line of curvature for both Σ_1 and Σ_2 , and these surfaces cut one another orthogonally along the circle.

Since there is a surface Σ_1 for each curve $v = \text{const}$ on S_1 and a surface Σ_2 for each $u = \text{const}$, the circles of a cyclic system and the orthogonal surfaces may be looked upon as a system of three families of surfaces such that through each point in space there passes a surface of each family. Moreover, each of these three surfaces meets the other two orthogonally, and each curve of intersection is a line of curvature on both surfaces. We have seen (§ 96) that the confocal quadrics form such a system of surfaces, and another example is afforded by a family of parallel surfaces and the developables of the congruence of normals to these surfaces.

When three families of surfaces are so constituted that through each point of space there passes a surface of each family and each of the three surfaces meets the other two orthogonally, they are

in a *triply orthogonal system*. In the preceding examples of intersection of any two surfaces is a line of curvature Dupin showed that this is a property of all triply orthogonal systems. We shall prove this theorem in the next section.

General equations. Theorem of Dupin. The simplest example of a triply orthogonal system is afforded by the planes parallel to the coordinate planes. The equations of the system are

$$x = u_1, \quad y = u_2, \quad z = u_3,$$

u_1, u_2, u_3 are parameters. Evidently the values of these parameters corresponding to the planes through a point are the rectangular coordinates of the point. In like manner, the surfaces of any triply orthogonal system may be determined by their equations, and the values of the three parameters for the surfaces through a point constitute the *curvilinear coordinates* of the point. Between the latter and the rectangular coordinates we obtain equations of the form

$$x = f_1(u_1, u_2, u_3), \quad y = f_2(u_1, u_2, u_3), \quad z = f_3(u_1, u_2, u_3),$$

where the functions f are analytic in the domain considered. An example of this is afforded by formulas (VII, 8), which define space by a system of confocal quadrics.

For that the system be orthogonal it is necessary and sufficient that these functions satisfy the three conditions

$$\frac{\partial x}{\partial u_1} \frac{\partial x}{\partial u_2} = 0, \quad \sum \frac{\partial x}{\partial u_i} \frac{\partial x}{\partial u_j} = 0, \quad \sum \frac{\partial x}{\partial u_i} \frac{\partial x}{\partial u_j} = 0$$

of the surfaces $u_i = \text{const}$ is defined by (1) when u_i is a constant value.

Linear element of space at a point we mean the linear element of any curve through it. This is

$$ds^2 = dx^2 + dy^2 + dz^2,$$

in consequence of (2), may be written in the parametric form

$$ds^2 = H_1^2 du_1^2 + H_2^2 du_2^2 + H_3^2 du_3^2,$$

$$H_1^2 = \sum \left(\frac{\partial x}{\partial u_1} \right)^2, \quad H_2^2 = \sum \left(\frac{\partial y}{\partial u_2} \right)^2, \quad H_3^2 = \sum \left(\frac{\partial z}{\partial u_3} \right)^2$$

defined, the functions H_1, H_2, H_3 are real and we shall assume that they are positive.

From (3) we have at once the linear element of any of the surfaces of the system. For instance, the linear element of a surface $u_3 = \text{const}$ is

$$ds_3^2 = H_1^2 du_1^2 + H_2^2 du_2^2$$

Now we shall find that the second quadratic forms of these surfaces are expressible in terms of the functions H and their derivatives.

If X_i, Y_i, Z_i denote the direction-cosines of the normals to the surfaces $u_i = \text{const}$, we have

$$(5) \quad X_i = \frac{1}{H_i} \frac{\partial x}{\partial u_i}, \quad Y_i = \frac{1}{H_i} \frac{\partial y}{\partial u_i}, \quad Z_i = \frac{1}{H_i} \frac{\partial z}{\partial u_i}$$

We choose the axes such that

$$(6) \quad \begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix} = +1$$

Consequence of (5) the second fundamental coefficients of a surface $u_i = \text{const}$ are defined by

$$D_i = \frac{1}{H_i} \sum \frac{\partial x}{\partial u_i} \frac{\partial^2 x}{\partial u_i \partial u_\kappa}, \quad D'_i = \frac{1}{H_i} \sum \frac{\partial x}{\partial u_i} \frac{\partial^2 x}{\partial u_i \partial u_\kappa \partial u_l}, \quad D''_i = \frac{1}{H_i} \sum \frac{\partial x}{\partial u_i} \frac{\partial^2 x}{\partial u_i \partial u_l^2},$$

where ι, κ, l take the values 1, 2, 3 in cyclic order, and the sign Σ refers to the summation of terms in x, y, z , as formerly. In order to evaluate these expressions we differentiate equations (2) with respect to u_3, u_1, u_2 respectively. This gives

$$\sum \frac{\partial x}{\partial u_1} \frac{\partial^2 x}{\partial u_2 \partial u_3} + \sum \frac{\partial x}{\partial u_2} \frac{\partial^2 x}{\partial u_3 \partial u_1} = 0,$$

$$\sum \frac{\partial x}{\partial u_2} \frac{\partial^2 x}{\partial u_1 \partial u_3} + \sum \frac{\partial x}{\partial u_3} \frac{\partial^2 x}{\partial u_1 \partial u_2} = 0,$$

$$\sum \frac{\partial x}{\partial u_3} \frac{\partial^2 x}{\partial u_1 \partial u_2} + \sum \frac{\partial x}{\partial u_1} \frac{\partial^2 x}{\partial u_2 \partial u_3} = 0$$

If each of these equations be subtracted from one half of the sum of the three, we have

$$\sum \frac{\partial x}{\partial u_1} \frac{\partial^2 x}{\partial u_2 \partial u_3} = 0, \quad \sum \frac{\partial x}{\partial u_2} \frac{\partial^2 x}{\partial u_3 \partial u_1} = 0, \quad \sum \frac{\partial x}{\partial u_3} \frac{\partial^2 x}{\partial u_1 \partial u_2} = 0,$$

consequently $D'_i = 0$

first and third of (2) be differentiated with respect to u_2 respectively, and the second and third of (4) with respect to u_1 , we have

$$\sum \frac{\partial x}{\partial u_1} \frac{\partial^2 x}{\partial u_2^2} = - \sum \frac{\partial x}{\partial u_2} \frac{\partial^2 x}{\partial u_1 \partial u_2} = - H_2 \frac{\partial H_2}{\partial u_1},$$

$$\sum \frac{\partial x}{\partial u_1} \frac{\partial^2 x}{\partial u_3^2} = - \sum \frac{\partial x}{\partial u_3} \frac{\partial^2 x}{\partial u_1 \partial u_3} = - H_3 \frac{\partial H_3}{\partial u_1}$$

we have

$$D_1 = - \frac{H_2}{H_1} \frac{\partial H_2}{\partial u_1}, \quad D_1'' = - \frac{H_3}{H_1} \frac{\partial H_3}{\partial u_1}$$

proceeding in like manner, we find the expressions for the other two. We write as follows

$$D_1 = - \frac{H_2}{H_1} \frac{\partial H_2}{\partial u_1}, \quad D_1' = 0, \quad D_1'' = - \frac{H_3}{H_1} \frac{\partial H_3}{\partial u_1},$$

κ, l take the values 1, 2, 3 in cyclic order. From the preceding equations and the fact that the parametric system on each surface is orthogonal, follows the theorem of Dupin

Surfaces of a triply orthogonal system meet one another at right angles at every point.

Equations of Lamé By means of these results we find the conditions to be satisfied by H_1, H_2, H_3 , in order that (3) may be the element of space referred to a triply orthogonal system. For each surface the Codazzi and Gauss equations must be satisfied. When the above values are substituted in these equations, we find the following six equations which it is necessary and sufficient that the functions H satisfy

$$\frac{\partial H_i}{\partial u_\kappa \partial u_l} = \frac{1}{H_\kappa} \frac{\partial H_\kappa}{\partial u_l} \frac{\partial H_i}{\partial u_\kappa} + \frac{1}{H_l} \frac{\partial H_l}{\partial u_\kappa} \frac{\partial H_i}{\partial u_l},$$

$$\frac{\partial}{\partial u_l} \left(\frac{1}{H_\kappa} \frac{\partial H_\kappa}{\partial u_l} \right) + \frac{\partial}{\partial u_\kappa} \left(\frac{1}{H_l} \frac{\partial H_l}{\partial u_\kappa} \right) + \frac{1}{H_i^2} \frac{\partial H_i}{\partial u_l} \frac{\partial H_i}{\partial u_\kappa} = 0,$$

κ, l take the values 1, 2, 3 in cyclic order. These are the *equations of Lamé*, being named for the geometer who first discovered them.*

For each of the surfaces there is a system of equations of the form (V, 16). When the values from (7) are substituted in these equations we have

$$\left\{ \begin{aligned} \frac{\partial X_i}{\partial u_i} &= -\frac{1}{H_\kappa} \frac{\partial H_i}{\partial u_\kappa} X_\kappa - \frac{1}{H_i} \frac{\partial H_i}{\partial u_i} X_i, \\ \frac{\partial X_i}{\partial u_\kappa} &= \frac{1}{H_i} \frac{\partial H_\kappa}{\partial u_i} X_\kappa, \quad \frac{\partial X_i}{\partial u_i} = \frac{1}{H_i} \frac{\partial H_i}{\partial u_i} X_i \end{aligned} \right.$$

Recalling the results of § 65, we have that each set of solutions of equations (8), (9) determine a triply orthogonal system, unique within a motion in space. In order to obtain the coordinates of space referred to this system, we must find nine functions X_i, Y_i, Z_i which satisfy (10) and

$$\sum X_i^2 = 1, \quad \sum X_i X_\kappa = 0 \quad (i \neq \kappa)$$

Then the coordinates of space are given by quadratures of the form

$$x = \int H_1 X_1 du_1 + H_2 X_2 du_2 + H_3 X_3 du_3$$

If $\rho_{i\kappa}$ denotes the principal radius of a surface $u_i = \text{const}$ in the direction of the curve of parameter u_κ , we have, from (7),

$$(11) \quad \frac{1}{\rho_{i\kappa}} = -\frac{1}{H_i H_\kappa} \frac{\partial H_\kappa}{\partial u_i}$$

Let ρ_1 denote the radius of first curvature of a curve of parameter u_1 . In accordance with § 49 we let $\bar{\omega}_1$ and $\bar{\omega}_1 - \pi/2$ denote the angles which the tangents to the curves of parameter u_2 and u_3 respectively through the given point make, in the positive sense, with the positive direction of the principal normal of the curve of parameter u_1 . Hence, by (IV, 16), we have

$$(12) \quad \frac{\cos \bar{\omega}_1}{\rho_1} = \frac{1}{\rho_{31}}, \quad \frac{\sin \bar{\omega}_1}{\rho_1} = \frac{1}{\rho_{21}}$$

From these equations and similar ones for curves of parameter u_2 and u_3 , we deduce the relations

$$(13) \quad \frac{1}{\rho_1^2} = \frac{1}{\rho_{\kappa 1}^2} + \frac{1}{\rho_{h 1}^2}, \quad \tan \bar{\omega}_1 = \frac{\rho_{h 1}}{\rho_{\kappa 1}},$$

κ, l take the values 1, 2, 3 in cyclic order. Moreover, since metric curves are lines of curvature, it follows from (§ 59) torsion of a curve of parameter u , is

$$\frac{1}{\tau} = \frac{1}{H} \frac{\partial \bar{\omega}_1}{\partial u_1}$$

Triple systems containing one family of surfaces of revolution

family of plane curves and their orthogonal trajectories, can be revolved about a line of the plane as an axis, the lines of surfaces of revolution thus generated, and the planes the axis, form a triply orthogonal system. We inquire there are any other triple systems containing a family of revolution.

Since that the surfaces $u_3 = \text{const}$ of a triple system are surfaces of revolution, and that the curves $u_2 = \text{const}$ upon them are meridians. Since the latter are geodesics, we must have

$$\frac{\partial H_1}{\partial u_2} = 0$$

(8) it follows that either

$$\frac{\partial H_1}{\partial u_3} = 0, \quad \text{or} \quad \frac{\partial H_3}{\partial u} = 0$$

In the first case it follows from (11) that $1/\rho_{31} = 0$. Consequently, surfaces of revolution $u_3 = \text{const}$ are developables, that is, either cylinders or circular cones. Furthermore, from (15) and we have $1/\rho_1 = 0$, so that the surfaces $u = \text{const}$ also are developables, and in addition we have, from (13), that $1/\rho_1 = 0$, that lines of parameter u_1 are straight lines and consequently curves $u_1 = \text{const}$ are parallel. The latter are planes when surfaces $u_1 = \text{const}$ are cylinders, and surfaces with circular curvature when $u_1 = \text{const}$ are circular cones. Conversely, by the theorem of Darboux (§ 187) and from § 132, it follows that a system of circular cylinders with parallel generators, or only of circular cones whose axes are tangent to the locus of vertex, leads to a triple system of the kind sought. Consider now the second case, namely

$$\frac{\partial H_1}{\partial u_2} = 0, \quad \frac{\partial H_3}{\partial u_2} = 0$$

From (11) we find that $1/\rho_{21} = 0$, and $1/\rho_{23} = 0$, consequently the surfaces $u_2 = \text{const.}$ are planes. Since these are the planes of the meridians, it follows that the axes of the surfaces coincide, and consequently the case cited at the beginning of this section is the only one for nondevelopable surfaces.

§185 Triple systems of Bianchi and of Weingarten In §119 it was found that all the Bianchi transforms of a given pseudospherical surface are pseudospherical surfaces of the same total curvature, and that they are the orthogonal surfaces of a cyclic system of circles of constant radius. Hence the totality of these circles and surfaces constitutes a triply orthogonal system, such that the surfaces in one family are pseudospherical. As systems of this sort were first considered by Ribaucour (cf §119), they are called the *triple systems of Ribaucour*. We proceed to the consideration of all triple systems such that the surfaces of one family are pseudospherical. These systems were first studied by Bianchi,* and consequently Darboux† has called them the *systems of Bianchi*.

From §119 it follows that the parameters of the lines of curvature of a pseudospherical surface of curvature $-1/a^2$ can be so chosen that the linear element takes the form

$$(16) \quad ds^2 = \cos^2 \omega \, du^2 + \sin^2 \omega \, dv^2,$$

where ω is a solution of the equation

$$(17) \quad \frac{\partial^2 \omega}{\partial u^2} - \frac{\partial^2 \omega}{\partial v^2} = \frac{\sin \omega \cos \omega}{a^4}$$

In this case the principal radii are given by

$$(18) \quad \frac{1}{\rho_1} = -\frac{\tan \omega}{a}, \quad \frac{1}{\rho_2} = \frac{\cot \omega}{a}$$

In general the total curvature of the pseudospherical surfaces of a system of Bianchi varies with the surfaces. If the surfaces $u_3 = \text{const.}$ are the pseudospherical surfaces, we may write the curvature in the form $-1/U_3^2$, where U_3 is a function of u_3 alone.

**Annali* Ser. 2, Vol. XIII (1885), pp. 177-234, Vol. XIV (1886), pp. 115-130, *Lezioni*, Vol. II, chap. xxvii.

† *Leçons sur les systèmes orthogonaux et les coordonnées curvilignes*, pp. 308-323, Paris, 1898.

rdance with (11) and (18) we put

$$\begin{cases} \frac{1}{\rho_{31}} = -\frac{1}{H_3 H_1} \frac{\partial H_1}{\partial u_3} = \frac{\tan \omega}{U_3}, \\ \frac{1}{\rho_{32}} = -\frac{1}{H_3 H_2} \frac{\partial H_2}{\partial u_3} = -\frac{\cot \omega}{U_3} \end{cases}$$

values of $\frac{\partial H_1}{\partial u_3}$ and $\frac{\partial H_2}{\partial u_3}$ be substituted in equations (8) for 1 and 2 respectively, we obtain

$$\frac{1}{H_1} \frac{\partial H_1}{\partial u_2} = -\tan \omega \frac{\partial \omega}{\partial u_2}, \quad \frac{1}{H_2} \frac{\partial H_2}{\partial u_1} = \cot \omega \frac{\partial \omega}{\partial u_1}$$

se equations we have, by integration,

$$H_1 = \phi_{13} \cos \omega, \quad H_2 = \phi_{23} \sin \omega,$$

and ϕ_{23} are functions independent of u_2 and u_1 respectively show that both of them are independent of u_3 the values of H_1 and H_2 from (20) are substituted in (19), respectively

$$\begin{cases} H_3 = U_3 \cot \omega \left(\tan \omega \frac{\partial \omega}{\partial u_3} - \frac{\partial \log \phi_{13}}{\partial u_3} \right), \\ H_3 = U_3 \tan \omega \left(\cot \omega \frac{\partial \omega}{\partial u_3} + \frac{\partial \log \phi_{23}}{\partial u_3} \right) \end{cases}$$

se equations it follows that

$$\cot \omega \frac{\partial \log \phi_{13}}{\partial u_3} + \tan \omega \frac{\partial \log \phi_{23}}{\partial u_3} = 0$$

nless ϕ_{13} and ϕ_{23} are independent of u_3 , $\tan \omega$ is equal to of a function of u_1 and u_2 and of a function of u and u_3 nsider the latter case and study for the moment a partic- ace $u_1 = u$ By the change of parameters

$$\phi_{13}(u_1, c) du_1 = du, \quad \phi_{23}(u_2, c) du_2 = dv,$$

r element of the surface reduces to (16), and (22) becomes

$$\tan \omega = \frac{U}{V},$$

and V are functions of u and v respectively When this substituted in (17), we obtain

$$\left(\frac{U''}{U} + \frac{V''}{V} \right) (U^2 + V^2) = \frac{U^2 + V^2}{a^2} + 2 U'^2 + 2 V'^2$$

If this equation be differentiated successively with respect to u and v , we find

$$\left(\frac{U''}{U}\right)' \frac{1}{U U'} + \left(\frac{V''}{V}\right)' \frac{1}{V V'} = 0,$$

unless U' or V' is equal to zero. From this it follows that

$$\left(\frac{U''}{U}\right)' = 4 \kappa U U', \quad \left(\frac{V''}{V}\right)' = -4 \kappa V V',$$

where κ denotes a constant. Integrating, we have

$$U'' = 2 \kappa U^3 + \alpha U, \quad V'' = -2 \kappa V^3 + \beta V,$$

α and β being constants, and another integration gives

$$U'^2 = \kappa U^4 + \alpha U^2 + \gamma, \quad V'^2 = -\kappa V^4 + \beta V^2 + \delta$$

When these expressions are substituted in (23), we find

$$\left(\beta - \alpha - \frac{1}{\alpha^2}\right) U^2 + \left(\alpha - \beta - \frac{1}{\alpha^2}\right) V^2 - 2\gamma - 2\delta = 0$$

This condition can be satisfied only when the curvature is zero. Hence U' or V' must be zero, that is, ω must be a function of u or v alone. In this case the surface is a surface of revolution. In accordance with § 184 a triple system of Bianchi arises from an infinity of pseudospherical surfaces of revolution with the same axis.

When exception is made of this case, the functions ϕ_{13} and ϕ_{23} in (20) are independent of u_3 . Hence the parameters of the systems may be chosen so that we have

$$(24) \quad H_1 = \cos \omega, \quad H_2 = \sin \omega, \quad H_3 = U_3 \frac{\partial \omega}{\partial u_3}$$

When these values are substituted in the six equations (8), (9), they reduce to the four equations

$$(25) \quad \begin{cases} \frac{\partial^2 \omega}{\partial u_1^2} - \frac{\partial^2 \omega}{\partial u_2^2} - \frac{\sin \omega \cos \omega}{U_3^2} = 0, \\ \frac{\partial^3 \omega}{\partial u_1 \partial u_2 \partial u_3} - \cot \omega \frac{\partial \omega}{\partial u_1} \frac{\partial^2 \omega}{\partial u \partial u_3} + \tan \omega \frac{\partial \omega}{\partial u_2} \frac{\partial^2 \omega}{\partial u_1 \partial u_3} = 0, \\ \frac{\partial}{\partial u_1} \left(\frac{1}{\cos \omega} \frac{\partial^2 \omega}{\partial u_1 \partial u_3} \right) - \frac{1}{U_3} \frac{\partial}{\partial u_3} \left(\frac{\sin \omega}{U_3} \right) - \frac{1}{\sin \omega} \frac{\partial \omega}{\partial u_2} \frac{\partial^2 \omega}{\partial u_2 \partial u_3} = 0, \\ \frac{\partial}{\partial u_2} \left(\frac{1}{\sin \omega} \frac{\partial^2 \omega}{\partial u_2 \partial u_3} \right) + \frac{1}{U_3} \frac{\partial}{\partial u_3} \left(\frac{\cos \omega}{U_3} \right) + \frac{1}{\cos \omega} \frac{\partial \omega}{\partial u_1} \frac{\partial^2 \omega}{\partial u_1 \partial u_3} = 0 \end{cases}$$

has inquired into the generality of the solution of these equations, and he has found that the general solution involves arbitrary functions of a single variable. We shall give a proof of this fact, but refer the reader to the investigation of Darboux*.

We now turn to the consideration of the particular case where the curvature of all the pseudospherical surfaces is the same, γ may be taken to be -1 without any loss of generality. These systems of this sort were first discussed by Weingarten, and Bianchi in calling them *systems of Weingarten*. Of these are triple systems of Ribaucour.

In this case we have $U_3 = 1$, so that the linear element of space is

$$ds^2 = \cos^2 \omega du_1^2 + \sin^2 \omega du_2^2 + \left(\frac{\partial \omega}{\partial u_3} \right)^2 du_3^2$$

The second of equations (25) may be written in either of

$$\begin{cases} \frac{\partial}{\partial u_1} \left(\frac{1}{\sin \omega} \frac{\partial^2 \omega}{\partial u_1 \partial u_3} \right) = - \frac{1}{\cos \omega} \frac{\partial \omega}{\partial u_2} \frac{\partial^2 \omega}{\partial u_1 \partial u_3}, \\ \frac{\partial}{\partial u_2} \left(\frac{1}{\cos \omega} \frac{\partial^2 \omega}{\partial u_1 \partial u_3} \right) = \frac{1}{\sin \omega} \frac{\partial \omega}{\partial u_1} \frac{\partial^2 \omega}{\partial u_2 \partial u_3}, \end{cases}$$

$$\Phi = \left(\frac{1}{\cos \omega} \frac{\partial^2 \omega}{\partial u_1 \partial u_3} \right)^2 + \left(\frac{1}{\sin \omega} \frac{\partial^2 \omega}{\partial u_2 \partial u_3} \right)^2 - \left(\frac{\partial \omega}{\partial u_3} \right)^2,$$

from the last two of (25) and from (27) that

$$\frac{\partial \Phi}{\partial u_1} = 0, \quad \frac{\partial \Phi}{\partial u_2} = 0$$

is a function of u_3 alone. But by changing the parameter u_3 to c , an operation which will not affect the form of (26), we can make c a constant value, say c . Consequently we have

$$\left(\frac{1}{\cos \omega} \frac{\partial^2 \omega}{\partial u_1 \partial u_3} \right)^2 + \left(\frac{1}{\sin \omega} \frac{\partial^2 \omega}{\partial u_2 \partial u_3} \right)^2 - \left(\frac{\partial \omega}{\partial u_3} \right)^2 = c$$

as shown† that equation (28) and the first of (25) are equivalent to the system (25), when $U_3 = 1$. Consequently the problem of the determination of triple systems of Weingarten is reduced to finding common solutions of these two equations

EXAMPLES

1 Show that the equations

$$x = r \cos u \cos v, \quad y = r \cos u \sin v, \quad z = r \sin u$$

define space referred to a triply orthogonal system

2 A necessary and sufficient condition that the surfaces $u_3 = \text{const}$ of a triply orthogonal system be parallel is that H_3 be a function of u_3 alone. What are the other surfaces $u_1 = \text{const}$, $u_2 = \text{const}$?

3 Two near-by surfaces $u_3 = \text{const}$ intercept equal segments on those orthogonal trajectories of the surfaces $u_3 = \text{const}$ which pass through a curve $H_3 = \text{const}$ on the former, on this account the curves $H_3 = \text{const}$ on the surfaces $u_3 = \text{const}$ are called *curves of equidistance*.

4 Let the surfaces $u_3 = \text{const}$ of a triple system be different positions of the same pseudosphere, obtained by translating the surface in the direction of its axis. Determine the character of the other surfaces of the system.

5 Derive the following results for a triple system of Weingarten

$$\Delta_1 \frac{\partial \omega}{\partial u_3} = c + \left(\frac{\partial \omega}{\partial u_3} \right)^2, \quad \rho_g = - \frac{\sqrt{c + \left(\frac{\partial \omega}{\partial u_3} \right)^2}}{\frac{\partial \omega}{\partial u_3}},$$

where the differential parameter is formed with respect to the linear element of ε surface $u_3 = \text{const}$, and ρ_g is the radius of geodesic curvature of a curve $\frac{\partial \omega}{\partial u_3} = \text{const}$ on this surface. Show that the curves of equidistance on the surfaces $u_3 = \text{const}$ are geodesic parallels of constant geodesic curvature.

6 Show that when c in (28) is equal to zero, the first curvature $1/\rho_3$ of the curves of parameter u_3 is constant and equal to unity, that equations similar to (12) become

$$\frac{\partial^2 \omega}{\partial u_2 \partial u_3} = - \sin \omega \cos \bar{\omega}_3 \frac{\partial \omega}{\partial u_3}, \quad \frac{\partial^2 \omega}{\partial u_1 \partial u_3} = - \cos \omega \sin \bar{\omega}_3 \frac{\partial \omega}{\partial u_3},$$

that if we put $\theta = \frac{\pi}{2} - \bar{\omega}_3$, the last two of equations (25), where $U_3 = 1$, may be written

$$\frac{\partial \theta}{\partial u_1} + \frac{\partial \omega}{\partial u_2} = \sin \theta \cos \omega, \quad \frac{\partial \theta}{\partial u_2} + \frac{\partial \omega}{\partial u_1} = - \cos \theta \sin \omega,$$

and that

$$\frac{\partial^2 \theta}{\partial u_1^2} - \frac{\partial^2 \theta}{\partial u_2^2} = \sin \theta \cos \theta, \quad \left(\frac{1}{\cos \theta} \frac{\partial^2 \theta}{\partial u_1 \partial u_3} \right)^2 + \left(\frac{1}{\sin \theta} \frac{\partial \theta}{\partial u_2 \partial u_3} \right)^2 = \left(\frac{\partial \theta}{\partial u_3} \right)^2$$

When $c = 0$ in (28) the system is said to be of *constant curvature*.

7 A necessary and sufficient condition that the curves of parameter u_3 of system of Weingarten be circles is that $\bar{\omega}_3$ be independent of u_3 . In this case (cf. Ex. 6) the surfaces $u_3 = \text{const}$ are the Bianchi transforms of the pseudo-spherical surface with the lineal element

$$ds^2 = \cos^2 \theta du^2 + \sin^2 \theta dv^2$$

Theorem of Ribaucour The following theorem is due to Ribaucour *

A family of surfaces of a triply orthogonal system and their principal trajectories, the osculating circles to the latter at their meeting with any surface of the family form a cyclic system

To prove this theorem we first derive the conditions to be satisfied by a system of circles orthogonal to a surface S so that they may form a cyclic system. Let the lines of curvature on S be parametric curves. Let the surface to the moving trihedral whose x - and y -axes are tangent to the curves $v = \text{const}$, $u = \text{const}$. We have (V, 63)

$$\xi_1 = \eta = p = q_1 = 0$$

Let ϕ be the angle which the plane of the circle through a point P makes with the corresponding xz -plane, θ the angle which the radius to a point P of the circle makes with its projection in the xz -plane, and R the radius of the circle, the coordinates of P in reference to the moving axes are

$$x = R(1 + \cos \theta) \cos \phi, \quad y = R(1 + \cos \theta) \sin \phi, \quad z = R \sin \theta$$

and the direction cosines of the tangent to the circle at P are

$$-\sin \theta \cos \phi, \quad -\sin \theta \sin \phi, \quad \cos \theta$$

To express the condition that every displacement of P must be perpendicular to this line, we have, from (29) and (V, 51),

$$\begin{aligned} & - \left[\sin \theta \left(\frac{1}{R} \frac{\partial R}{\partial u} + \frac{\xi \cos \phi}{R} \right) + q \cos \phi (1 + \cos \theta) \right] du \\ & - \left[\sin \theta \left(\frac{1}{R} \frac{\partial R}{\partial v} + \frac{\eta_1 \sin \phi}{R} \right) - p_1 \sin \phi (1 + \cos \theta) \right] dv = 0 \end{aligned}$$

The condition that this equation admit an integral is reducible to

$$\begin{aligned} & \left(\frac{\eta_1 \sin \phi}{R} \right) - \frac{\partial}{\partial v} \left(\frac{\xi \cos \phi}{R} \right) \Big] \sin \theta + \left[\frac{\sin \phi \cos \phi}{R} (\xi p_1 + \eta_1 q) \right. \\ & \left. - R \frac{\partial}{\partial u} \left(\frac{\sin \phi}{R} p_1 \right) - R \frac{\partial}{\partial v} \left(\frac{\cos \phi}{R} q \right) \right] (1 + \cos \theta) = 0 \end{aligned}$$

As remarked before (§ 174), if there are three surfaces orthogonal to a system of circles, the system is cyclic

The condition that it be cyclic is

$$(30) \quad \begin{cases} \frac{\partial}{\partial u} \left(\frac{\eta_1 \sin \phi}{R} \right) - \frac{\partial}{\partial v} \left(\frac{\xi \cos \phi}{R} \right) = 0, \\ \frac{\sin \phi \cos \phi}{R^2} (\xi p_1 + \eta_1 q) - \frac{\partial}{\partial u} \left(\frac{\sin \phi}{R} p_1 \right) - \frac{\partial}{\partial v} \left(\frac{\cos \phi}{R} q \right) = 0 \end{cases}$$

Since the principal radii of S are given by

$$(31) \quad \frac{1}{\rho_1} = -\frac{q}{\xi}, \quad \frac{1}{\rho_2} = \frac{p_1}{\eta_1},$$

the second of equations (30) reduces to the first when S is a sphere or a plane. Hence we have incidentally the theorem

A two parameter system of circles orthogonal to a sphere and to any other surface constitute a cyclic system

We return to the proof of the theorem of Ribaucour and apply the foregoing results to the system of osculating circles of the curves of parameter u_3 of an orthogonal system at their points of intersection with a surface $u_3 = \text{const}$

From equations similar to (12) we have, by (11),

$$\frac{\cos \phi}{R} = -\frac{1}{H_1 H_3} \frac{\partial H_1}{\partial u_1}, \quad \frac{\sin \phi}{R} = -\frac{1}{H_2 H_3} \frac{\partial H_1}{\partial u_2},$$

and the equations analogous to (31) are

$$\frac{1}{\rho_{31}} = -\frac{q}{H_1} = -\frac{1}{H_1 H_3} \frac{\partial H_1}{\partial u_3}, \quad \frac{1}{\rho_3} = \frac{p_1}{H_2} = -\frac{1}{H_2 H_3} \frac{\partial H_2}{\partial u_3}$$

When these values are substituted in equations (30) the first vanishes identically, likewise the second, in consequence of equations (8). Hence the theorem of Ribaucour is proved.*

187 Theorems of Darboux The question naturally arises whether any family of surfaces whatever forms part of a triply orthogonal system. This question will be answered with the aid of the following theorem of Darboux,† which we establish by his methods

A necessary and sufficient condition that two families of surfaces orthogonal to one another admit of a third family orthogonal to both is that the first two meet one another in lines of curvature

* For a geometrical proof the reader is referred to Darboux, *l. c.*, p. 77. † *L. c.*, pp. 6-8

two families of surfaces be defined by

$$\alpha(x, y, z) = a, \quad \beta(x, y, z) = b,$$

and b are the parameters. The condition of orthogonality is

$$\frac{\partial \alpha}{\partial x} \frac{\partial \beta}{\partial x} + \frac{\partial \alpha}{\partial y} \frac{\partial \beta}{\partial y} + \frac{\partial \alpha}{\partial z} \frac{\partial \beta}{\partial z} = 0$$

that a third family of surfaces exist orthogonal to the other two families, there must be a function $\gamma(x, y, z)$ such that the equations

$$\frac{\gamma}{x} + \frac{\partial \alpha}{\partial y} \frac{\partial \gamma}{\partial y} + \frac{\partial \alpha}{\partial z} \frac{\partial \gamma}{\partial z} = 0, \quad \frac{\partial \beta}{\partial x} \frac{\partial \gamma}{\partial x} + \frac{\partial \beta}{\partial y} \frac{\partial \gamma}{\partial y} + \frac{\partial \beta}{\partial z} \frac{\partial \gamma}{\partial z} = 0$$

dy, dz denote the projections on the axes of a displacement of a point on one of the surfaces $\gamma = \text{const}$, we must have

$$\begin{vmatrix} dx & dy & dz \\ \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} & \frac{\partial \alpha}{\partial z} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} & \frac{\partial \beta}{\partial z} \end{vmatrix} = 0$$

relation is of the form (XIII, 9). The condition (XIII, 10) is the limit of an integral involving a parameter is

$$\sum_{\alpha, \beta, \gamma} \left[\frac{\partial \beta}{\partial x} \frac{\partial^2 \alpha}{\partial z^2} + \frac{\partial \alpha}{\partial z} \frac{\partial^2 \beta}{\partial x \partial z} - \frac{\partial \alpha}{\partial x} \frac{\partial^2 \beta}{\partial z^2} - \frac{\partial \beta}{\partial z} \frac{\partial^2 \alpha}{\partial x \partial z} - \frac{\partial \beta}{\partial y} \frac{\partial^2 \alpha}{\partial x \partial y} - \frac{\partial \alpha}{\partial x} \frac{\partial^2 \beta}{\partial y^2} + \frac{\partial \alpha}{\partial y} \frac{\partial^2 \beta}{\partial x \partial y} + \frac{\partial \beta}{\partial x} \frac{\partial^2 \alpha}{\partial y^2} \right] = 0,$$

indicates the sum of the three terms obtained by permuting x, y, z in this expression. If we add to this equation the identity

$$\sum_{\alpha, \beta, \gamma} \frac{\partial(\alpha, \beta)}{\partial(y, z)} \left[\frac{\partial \alpha}{\partial x} \sum_{\beta, \gamma} \frac{\partial^2 \beta}{\partial x} - \frac{\partial \beta}{\partial x} \sum_{\alpha, \gamma} \frac{\partial^2 \alpha}{\partial x} \right] = 0,$$

the equation may be written in the form

$$\begin{vmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \beta}{\partial x} & \delta\left(\alpha, \frac{\partial \beta}{\partial x}\right) - \delta\left(\beta, \frac{\partial \alpha}{\partial x}\right) \\ \frac{\partial \alpha}{\partial y} & \frac{\partial \beta}{\partial y} & \delta\left(\alpha, \frac{\partial \beta}{\partial y}\right) - \delta\left(\beta, \frac{\partial \alpha}{\partial y}\right) \\ \frac{\partial \alpha}{\partial z} & \frac{\partial \beta}{\partial z} & \delta\left(\alpha, \frac{\partial \beta}{\partial z}\right) - \delta\left(\beta, \frac{\partial \alpha}{\partial z}\right) \end{vmatrix} = 0,$$

where, for the sake of brevity, we have introduced the symbol $\delta(\theta, \phi)$, defined by

$$\delta(\theta, \phi) = \frac{\partial \theta}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial \theta}{\partial y} \frac{\partial \phi}{\partial y} + \frac{\partial \theta}{\partial z} \frac{\partial \phi}{\partial z}$$

If equation (33) be differentiated with respect to x , the result may be written

$$\delta\left(\alpha, \frac{\partial \beta}{\partial x}\right) + \delta\left(\beta, \frac{\partial \alpha}{\partial x}\right) = 0$$

Consequently equation (34) is reducible to

$$(35) \quad \begin{vmatrix} \frac{\partial \alpha}{\partial x} & \frac{\partial \beta}{\partial x} & \delta\left(\alpha, \frac{\partial \beta}{\partial x}\right) \\ \frac{\partial \alpha}{\partial y} & \frac{\partial \beta}{\partial y} & \delta\left(\alpha, \frac{\partial \beta}{\partial y}\right) \\ \frac{\partial \alpha}{\partial z} & \frac{\partial \beta}{\partial z} & \delta\left(\alpha, \frac{\partial \beta}{\partial z}\right) \end{vmatrix} = 0,$$

which is therefore the condition upon α and β in order that the desired function γ exist

A displacement along a curve orthogonal to the surfaces $\alpha = \text{const}$ is given by

$$\frac{dx}{\frac{\partial \alpha}{\partial x}} = \frac{dy}{\frac{\partial \alpha}{\partial y}} = \frac{dz}{\frac{\partial \alpha}{\partial z}}$$

Such a curve lies upon a surface $\beta = \text{const}$, and since, by (35), it satisfies the condition

$$\begin{vmatrix} dx & \frac{\partial \beta}{\partial x} & d\frac{\partial \beta}{\partial x} \\ dy & \frac{\partial \beta}{\partial y} & d\frac{\partial \beta}{\partial y} \\ dz & \frac{\partial \beta}{\partial z} & d\frac{\partial \beta}{\partial z} \end{vmatrix} = 0,$$

it is a line of curvature on the surface (cf Ex 3, p 247). Hence the curves of intersection of the surfaces $\alpha = \text{const}$, $\beta = \text{const}$, being the orthogonal trajectories of the above curves, are lines of curvature on the surfaces $\beta = \text{const}$. And by Joachimsthal's theorem (§ 59) they are lines of curvature on the surfaces $\alpha = \text{const}$ also. Having thus established the theorem of Darboux, we are in a position to answer the question at the beginning of this section

a family of surfaces $\alpha = \text{const}$, the lines of curvature in any family form a congruence of curves which must admit a family of orthogonal surfaces, if the surfaces $\alpha = \text{const}$ are to form part of an orthogonal system. If this condition is satisfied, then, according to the theorem of Darboux, there is a third family of surfaces together with the other two form an orthogonal system.

X_1, Y_1, Z_1 denote the direction-cosines of the tangents to the lines of curvature in one family on the surfaces $\alpha = \text{const}$, the condition that there be a family of surfaces orthogonal to the first family is that the equation

$$X_1 dx + Y_1 dy + Z_1 dz = 0$$

be an integral involving a parameter. The condition for this is

$$X_1 \left(\frac{\partial Z_1}{\partial y} - \frac{\partial Y_1}{\partial z} \right) + Y_1 \left(\frac{\partial X_1}{\partial z} - \frac{\partial Z_1}{\partial x} \right) + Z_1 \left(\frac{\partial Y_1}{\partial x} - \frac{\partial X_1}{\partial y} \right) = 0$$

in order to find X_1, Y_1, Z_1 we remark that since they are the direction-cosines of the tangents to a line of curvature we must have

$$\frac{\partial X}{\partial x} X_1 + \frac{\partial X}{\partial y} Y_1 + \frac{\partial X}{\partial z} Z_1 = \lambda X_1,$$

similar equations in Y, Z , where the function λ is a factor of proportionality to be determined and X, Y, Z are the direction-cosines of the normal to the surface $\alpha = \text{const}$. Hence, if the surfaces are defined by $\alpha = \text{const}$, the functions X_1, Y_1, Z_1 are expressible in terms of the first and second derivatives of α , and the condition (36) is of the third order in these derivatives. Therefore, we have the theorem of Darboux*:

determination of all triply orthogonal systems requires the solution of a partial differential equation of the third order

Darboux has given the name *family of Lamé* to a family of surfaces which forms part of a triply orthogonal system.

Transformation of Combescure We close our study of triply orthogonal surfaces with an exposition of the *transformation of Combescure*,† by means of which from a given orthogonal system can be obtained such that the normals to the surfaces of the first system are parallel to the normals to the corresponding surfaces of the other system at corresponding points.

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We make use of a set of functions β_{ik} , introduced by Darboux* in his development of a similar transformation in space of n dimensions. By definition

$$\beta_{ik} = \frac{1}{H_i} \frac{\partial H_k}{\partial u_i} \quad (i \neq k)$$

In terms of these functions equations (8), (9) are expressible in

$$\frac{\partial \beta_{ik}}{\partial u_i} = \beta_{ii} \beta_{ik}, \quad \frac{\partial \beta_{ik}}{\partial u_i} + \frac{\partial \beta_{ki}}{\partial u_k} + \beta_{ii} \beta_{ik} = 0,$$

which become

$$\frac{\partial X_i}{\partial u_i} = -\beta_{ki} X_k - \beta_{ii} X_i, \quad \frac{\partial X_i}{\partial u_k} = \beta_{ik} X_k$$

Conditions (37), (38) are the necessary and sufficient conditions that the expression

$$X_1 H_1 du_1 + X_2 H_2 du_2 + X_3 H_3 du_3$$

be an exact differential. From their form it is seen that if we have another set of functions H'_1, H'_2, H'_3 satisfying the six conditions

$$(39) \quad \beta_{ik} = \frac{1}{H'_i} \frac{\partial H'_k}{\partial u_i},$$

where the functions β_{ik} have the same values as for the given system, the expression

$$X_1 H'_1 du_1 + X_2 H'_2 du_2 + X_3 H'_3 du_3,$$

and similar ones in Y, Z , are exact differentials, and so by quadratures we obtain an orthogonal system possessing the desired property.

In order to ascertain the analytical character of this problem, we eliminate H'_2 and H'_3 from equations (39) and obtain the three equations

$$\begin{aligned} \frac{\partial^2 H'_1}{\partial u_1 \partial u_1} &= \frac{1}{\beta_{11}} \frac{\partial \beta_{11}}{\partial u_1} \frac{\partial H'_1}{\partial u_1} + \beta_{21} \beta_{12} H'_1, \\ \frac{\partial^2 H'_1}{\partial u_2 \partial u_3} &= \frac{\beta_{31} \beta_{23}}{\beta_{21}} \frac{\partial H'_1}{\partial u_3} + \frac{\beta_{21} \beta_{32}}{\beta_{31}} \frac{\partial H'_1}{\partial u_2}, \\ \frac{\partial^2 H'_1}{\partial u_3 \partial u_1} &= \frac{1}{\beta_{31}} \frac{\partial \beta_{11}}{\partial u_1} \frac{\partial H'_1}{\partial u_3} + \beta_{31} \beta_{13} H'_1 \end{aligned}$$

The general integral of a system of equations of this kind involves three arbitrary functions each of a single parameter u_i . When one

integral, the corresponding values of H_1', H_2' are given directly

Hence we have the theorem

every triply orthogonal system there is associated an infinity of surfaces, depending upon three arbitrary functions, such that the surfaces of any two systems at corresponding points are orthogonal *

EXAMPLES

every system of Weingarten for which c in (28) is zero, the system of curvatures of parameter u_3 at points of a surface $u_3 = \text{const.}$ form a Ribaucour (§ 185)

The orthogonal trajectories of a family of Lamé are twisted curves of constant first curvature, the surfaces of the family are pseudospherical of equal curvature

every triply orthogonal system which is derived from a cyclic system by a transformation of Combescure possesses one family of plane orthogonal trajectories

The orthogonal trajectories of a family of Lamé are plane curves, the cyclic circles osculating these trajectories at the points of any surface of the system may be obtained from the given system by a transformation of Combescure. We determine the triply orthogonal systems which result from the application of the transformation of Combescure to a system of Ribaucour (§ 185)

GENERAL EXAMPLES

An inversion by reciprocal radius (§ 80) be effected upon a triply orthogonal system the resulting system will be of the same kind

We determine the character of the surfaces of the system obtained by an inversion of a system of Fx 1, § 185, and show that all the curves of intersection are circles. We establish the existence of a triply orthogonal system of spheres

A necessary and sufficient condition that the asymptotic lines correspond on the surfaces $u_3 = \text{const.}$ of a triply orthogonal system is that there exist a relation among

$$\phi_1 H_1^2 + \phi_2 H_2^2 + \phi_3 = 0,$$

where ϕ_1, ϕ_2, ϕ_3 are functions independent of u_3

When the condition of Fx 4 is satisfied, those orthogonal trajectories of the system $u_3 = \text{const.}$ which pass through points of an asymptotic line on a surface $u_3 = \text{const.}$ constitute a surface S which meets the surfaces $u_3 = \text{const.}$ in the asymptotic lines of the latter and geodesics on S

We show that the asymptotic lines correspond on the pseudospherical surfaces of the system of Bianchi

We show that there exist triply orthogonal systems for which the surfaces in one system are spherical, and that the parameters can be chosen so that

$$H_1 = \cosh \theta, \quad H_2 = \sinh \theta, \quad H_3 = U_3 \frac{\partial \theta}{\partial u_3}$$

the equations of Lamé for this case

every one-parameter family of spheres or planes is a family of Lamé

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9 In order to obtain the most general triply orthogonal system for which the surfaces in one family are planes, one need construct an orthogonal system of curves in a plane and allow the latter to roll over a developable surface, in which case the curves generate the other surfaces. When the developable is given, the determination of the system reduces to quadratures.

10 Show that the most general triply orthogonal system for which one family of Lamé consists of spheres passing through a point can be found by quadratures.

11 Show that a family of parallel surfaces is a family of Lamé.

12 Show that the triply orthogonal systems for which the curves of parameter u_3 are circles passing through a point can be found without quadrature.

13 By means of Ex. 6, § 185, show that for a system of Weingarten of constant curvature the principal normals to the curves of parameter u_3 at the points of meeting with a surface $u_3 = \text{const}$ form a normal pseudospherical congruence, and that the surfaces complementary to the surfaces $u_3 = \text{const}$ and their orthogonal trajectories constitute a system of Weingarten of constant curvature.

14 By means of Ex. 13 show that for a triple system arising from a system of Weingarten of constant curvature by a transformation of Combescure the osculating planes of the curves $u_3 = \text{const}$, at points of a surface $u_3 = \text{const}$, envelop a surface S of the same kind as this surface $u_3 = \text{const}$, and these surfaces S and their orthogonal trajectories constitute a system of the same kind as the one resulting from the Combescure transformation of the given system of Weingarten.

15 Show that a necessary condition that the curves of parameter u_1 of a triple system of Bianchi be plane is that ω satisfy also the conditions

$$\frac{\partial \omega}{\partial u_2} = \phi_{23} \sin \omega, \quad \frac{\partial \omega}{\partial u_1} = \phi_{13} \sin \omega,$$

where ϕ_{23} and ϕ_{13} are independent of u_1 and u_2 respectively (cf Ex. 5, p. 317). Show that if ϕ_{13} and ϕ_{23} satisfy the conditions

$$\left(\frac{\partial \phi_{13}}{\partial u_1}\right)^2 = \phi_{13}^4 + 2a\phi_{13}^2 + b, \quad \left(\frac{\partial \phi_{23}}{\partial u_2}\right)^2 = \left(\phi_{23}^2 + \frac{1}{U_3^2}\right)^2 + 2a\left(\phi_{23}^2 + \frac{1}{U_3^2}\right) + b,$$

where a and b are constants and U_3 is an arbitrary function of u_3 , the function ω , given by

$$\cos \omega = \frac{\frac{\partial \phi_{23}}{\partial u_2} - \frac{\partial \phi_{13}}{\partial u_1}}{\phi_{13}^2 - \phi_{23}^2 - \frac{1}{U_3^2}},$$

determines a triply orthogonal system of Bianchi of the kind sought.

16 When $U_3 = 1$ and ω is independent of u_2 , the first and fourth of equations (25) may be replaced by

$$\frac{\partial \omega}{\partial u_1} = \sin \omega$$

Show that for a value of ω satisfying this condition and the other equations (25) the expressions

$$H_1 = \cos \omega \left(\int \frac{\phi_3 du_3}{\sin \omega} + \phi_1 \right) - \int \frac{\phi_3 \cos \omega}{\sin \omega} du_3 + \phi_1,$$

$$H_2 = \sin \omega \left(\int \frac{\phi_3 du_3}{\sin \omega} + \phi_1 \right) - \int \phi_3 du_3 + \phi_2,$$

$$H_3 = \left(\int \frac{\phi_3 du_3}{\sin \omega} + \phi_1 \right) \frac{\partial \omega}{\partial u_3},$$

ϕ_2, ϕ_3 are functions of u_1, u_2, u_3 respectively, and the accent indicates a prime, define a triply orthogonal system for which the surfaces $u_3 = \text{const.}$ are spheres

Under what conditions do the functions

$$H_1 = U_2 \frac{\partial \omega}{\partial u_2}, \quad H_2 = - \left(\frac{U_2 \frac{\partial^2 \omega}{\partial u_2^2}}{\frac{\partial \omega}{\partial u_2}} + U_2' \right), \quad H_3 = - \frac{U_2 U_3}{\sin \omega} \frac{\partial^2 \omega}{\partial u_2 \partial u_3},$$

and U_2 are functions of u_2 and u_3 respectively, determine a triply orthogonal system arising from a triple system of Bianchi by a transformation of Combescure. Show that in this case the surfaces $u_2 = \text{const.}$ are spheres of radius U_2 , and curves of parameter u_2 in the system of Bianchi are plane or spherical

Prove that the equations

$$\begin{aligned} x &= A(u_1 - a)^{m_1} (u_2 - a)^{m_2} (u_3 - a)^{m_3}, \\ y &= B(u_1 - b)^{m_1} (u_2 - b)^{m_2} (u_3 - b)^{m_3}, \\ z &= C(u_1 - c)^{m_1} (u_2 - c)^{m_2} (u_3 - c)^{m_3}, \end{aligned}$$

B, C, a, b, c, m_1 are constants, define space referred to a triple system of surfaces such that each surface is cut by the surfaces of the other two families in two circles

Given a surface S and a sphere Σ , the circles orthogonal to both constitute a system, hence the locus of a point upon these circles which is in constant contact with the points of intersection with S and Σ is a surface S_1 orthogonal to S and Σ , S_1 may be looked upon as derived from S by a contact transformation which preserves lines of curvature, such a transformation preserves planes and spheres

When S of Ex. 19 is a cyclide of Dupin, so are the surfaces S_1 , and also the surface Σ which is the locus of the circles which meet S in any line of curvature, and the surfaces S_1 and Σ form a triple system of cyclides of Dupin

Given three functions U_i defined by

$$U_i = m_i u_i^2 + 2 n_i u_i + p_i, \quad (i = 1, 2, 3)$$

m_i, n_i, p_i are constants satisfying the conditions

$$\Sigma m_i = 0, \quad \Sigma n_i = 0, \quad \Sigma p_i = 0,$$

also the function

$$\begin{aligned} N &= \alpha_1(u_2 - u_3)\sqrt{U_1} + \alpha_2(u_3 - u_1)\sqrt{U_2} + \alpha_3(u_1 - u_2)\sqrt{U_3} \\ &\quad + \beta \Sigma m_i u_i + \gamma(p_1 u_2 u_3 + p_2 u_3 u_1 + p_3 u_1 u_2), \end{aligned}$$

α_i, β, γ are constants, determine under what condition the functions

$$H_1 = \frac{u_2 - u_3}{N\sqrt{U_1}}, \quad H_2 = \frac{u_3 - u_1}{N\sqrt{U_2}}, \quad H_3 = \frac{u_1 - u_2}{N\sqrt{U_3}}$$

define a triply orthogonal system. Show that all of the surfaces are isothermic, and that they are cyclides of Dupin

Determine whether there exist triply orthogonal systems of minimal surfaces

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* For references such as Transformation of Backlund *see* Backlund

$$10 \cdot 0 \cdot 0^2 \cdot 0^5$$

$$= u,$$

$$= \underline{x'^2 + y'^2 + z'^2 = 1}$$

$$\text{ent } \frac{X-x}{x'} = \frac{Y-y}{y'} = \frac{Z-z}{z'}$$

$$\text{Plane } (X-x)x' + (Y-y)y' + (Z-z)z' = 0$$

$$\text{Arc} = \frac{1}{\rho^2} = x''^2 + y''^2 + z''^2$$

$$\text{tting Plane: } \begin{vmatrix} X-x & Y-y & Z-z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}$$

$$\text{trihedron } \left\{ \begin{array}{l} \text{Tangent: } \frac{\text{Dir. Cos}}{\alpha, \beta, \gamma} = x' y' z' \\ \text{Binormal: } \lambda, \mu, \nu = \rho \begin{vmatrix} y'' z'' \\ y'' z'' \end{vmatrix}, \rho \begin{vmatrix} z'' x'' \\ z'' x'' \end{vmatrix} \\ \text{Prim. Normal: } l, m, n = \rho x'', \rho y'', \rho z'' \end{array} \right.$$

$$\text{tting Circle: Center} = X_0 Y_0 Z_0 = x + \rho l, y + \rho m, z + \rho n$$

$$\text{axis: } \frac{X-x-\rho l}{\lambda} = \frac{Y-y-\rho m}{\mu} = \frac{Z-z-\rho n}{\nu}$$

$$\text{on: } \frac{1}{\tau^2} = \lambda'^2 + \mu'^2 + \nu'^2$$

$$1 \quad \begin{vmatrix} x' & y' & z' \end{vmatrix}$$

Define transformation

$$\beta' = \frac{m}{\rho}, \quad \tau' = \frac{m}{\rho}$$

$$\left(\frac{1}{\rho} + \frac{\lambda}{\tau}\right), \quad m' = -\left(\frac{\beta}{\rho} + \frac{\mu}{\tau}\right), \quad n' = -\left(\frac{\gamma}{\rho} + \frac{\nu}{\tau}\right)$$

$$\mu' = \frac{m}{\tau}, \quad \nu' = \frac{m}{\tau}$$

ing Sphere:

$$a = x + \lambda \rho - \lambda \tau \rho'$$

$$b = y + m \rho - \mu \tau \rho'$$

$$c = z + n \rho - \nu \tau \rho'$$

$$r^2 = \rho^2 + \tau^2 \rho'^2$$

Surface

$$\text{Plane} \quad \begin{vmatrix} \xi-x & \eta-y & \zeta-z \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = 0$$

Initial quantities of First Order

$$\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$

$$\frac{x \partial x}{\partial u} + \frac{y \partial y}{\partial u} + \frac{z \partial z}{\partial u}$$

$$s = \frac{F}{\sqrt{EG}} \quad \sin \omega = \frac{F}{\sqrt{EG}} = \frac{F}{\sqrt{EG}}$$

unk of Area:

$$\sin \omega \sqrt{EG} du dv = H du dv$$

e between Two general curves C

$$J = \frac{E du du + F(du dv + dv du) + G dv dv}{ds ds}$$

tric Coordinate System

$$= \lambda^2 (du^2 + dv^2) \quad \text{if } u \text{ \& } v \text{ are an isotherm syst}$$

u, v are an isometric system any other tric system is of the form ϕ, ψ

$$\phi + i\psi = f(u + iv)$$

$$\phi + i\psi = f(u - iv)$$